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ELEMENTS

OF

DESCRIPTIVE GEOMETRY.

WITH APPLICATIONS TO

ISOMETRICAL DRAWING AND CAVALIER PROJECTION.

BY

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PREFACE.

Having been convinced, by a class-room experience of many years at the Stevens Institute of Technology, of the desirability of a text-book on Descriptive Geometry different in some respects from any previously existing, I have endeavored to produce a work suitable for use in colleges and scientific schools, and also by those who may wish to acquire some knowledge of the subject without the aid of an instructor. In the course of that experience many points have arisen, leading to original work embodied in this treatise; in the preparation of which, however, much benefit has been derived from reference to the works of Olivier, Jullien, Church, Warren, Watson, and others.

The study of Descriptive Geometry is not usually begun, nor should it ever be, until some familiarity with the ordinary operations of Mechanical Drawing has been attained. But when the former is taken up its identity with the latter should never be lost sight of, as it too often is: for this reason a departure has been made from the stereotyped methods of treatment, which in fact tend rather to conceal than to exhibit that identity.

At the outset considerable difficulty is often experienced in forming clear conceptions of the relations between abstract things, such as lines and planes, by the aid of orthographic projections only. The power of doing so is of course essential; and it is believed that the pictorial representations which have been introduced will be of assistance in acquiring it. But that power will be best developed, and greatly increased, by the instrumental construction of the problems—which indeed is absolutely necessary to the

attainment of such a mastery of the principles and processes as alone would be of any practical value.

As a hint to those who may choose to dispense with an instructor, it may be stated that at the Stevens Institute of Technology it is required that the diagrams shall be drawn with care, but not required that they shall be drawn in ink. Nor is the latter recommended; the time required to ink in one diagram can be better occupied in drawing another; moreover, work of this description affords the best of practice in neat, effective, and accurate pencilling,—an accomplishment which is becoming more and more important to the practical draughtsman.

C. W. MacCord.

Hoboken, New Jersey, *September* 23, 1895.

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1. Descriptive Geometry treats of the methods of making, with mathematical exactness, drawings for the representation not only of **geometrical magnitudes,** but of the **solutions** of problems relating to them.

2. This branch of science does not deal with the phenomena of binocular vision, and for its purposes the **eye** is regarded as a single point.

The **surface** upon which a drawing is made may be of any form, as cylindrical, in panoramic painting, or spherical, in decorating the interior of a dome. But in order to make correct drawings upon such surfaces, it is necessary to be thoroughly familiar with the methods of making them upon **planes,** which are usually employed; and to these our attention will be confined.

3. The **object** to be drawn may be placed between the eye and the plane, or the plane may be placed between the eye and the object. In either case, light is reflected from any point of the object to the eye in a right line; and the point in which that line, produced if necessary, pierces the plane, is the representation of

that point in the object from which it came. A sufficient number of such points being found, the outlines may be fully determined; and the drawing thus made will present to the eye, if placed in the position originally assigned to it, the same appearance as the actual contour of the object itself.

It may be said, then, that the representation of a point is found by *projecting* it along a right line passing through the eye. Such a line is called a **projecting line**, and all drawings thus made are technically called **projections**.

4. If the eye is at a finite distance, the drawing, on any surface, is called a **scenographic projection**. If made upon a vertical plane, against which the eye is directed *perpendicularly*, the drawing is said to be in **perspective**; the plane is then called the **picture plane**, and the projecting lines, which converge, are called **visual rays**.

If the eye is removed to an infinite distance, the projecting lines become parallel to each other and to the axis of vision. The plane upon which the drawing is made is called the **plane of projection**; it may be perpendicular to the **projecting lines**, in which case the drawing is called an **orthographic projection**; or it may cut them obliquely, and the drawing is then called an **oblique projection**.

5. Of these three systems of projection the second is the most simple and the most extensively used, and a knowledge of it is an essential preliminary to the study of the others. We proceed then at present to consider the methods of representing magnitudes and the solution of problems in **orthographic projection** only. Evidently the number of such projections, or views, necessary to the adequate representation of an object of three dimensions, will depend much on the form of the object itself. But, beginning with the least of geometrical magnitudes, the point, considered as a visible and material particle; it can be located in space by giving its distance from each of two fixed planes, and represented by its projections upon them.

6. The Principal Planes of Projection. The most simple and natural relation between two planes for this purpose, which is universally adopted, is that shown in Fig. 1; the one being horizontal, the other vertical. Moreover, these suffice for many, though by no means all, of the ordinary operations of descriptive

geometry; hence we may say that the principal planes of projection are

1. **The horizontal plane,** usually designated simply as **H** for brevity.

2. **The vertical plane,** usually designated simply as **V** for brevity.

These planes are supposed to extend indefinitely in each direction; they intersect in a line called the **ground line,** designated and referred to as **AB.** The eye is supposed to be at an infinitely remote point, **in front of the vertical plane** and **above the horizontal plane:** whence it is directed either perpendicularly against **V,** as

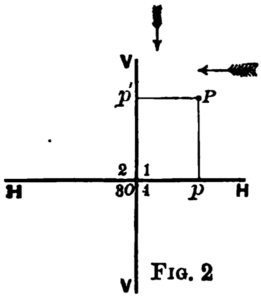


FIG. 2

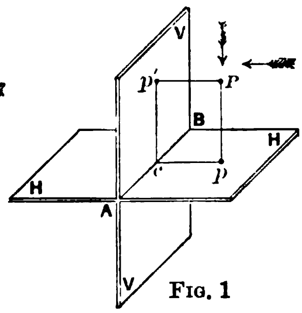


FIG. 1

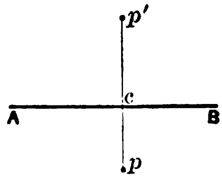


FIG. 3

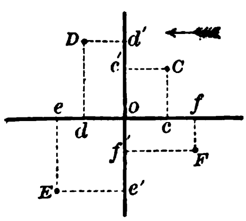


FIG. 4

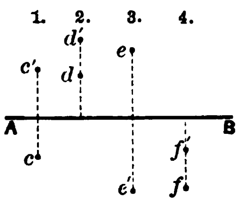


FIG. 5

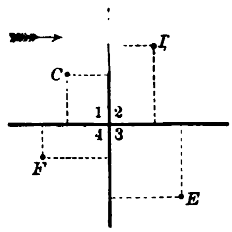


FIG. 6

shown by the horizontal arrow, or perpendicularly downward upon **H,** as shown by the vertical arrow.

7. The Four Angles. Fig. 1 is a pictorial representation of a model, such as can readily be made by cutting two cards, each through half its length, and "halving" them together. If this model be held so that the eye is directly in front of the point A, and looking in the direction AB, it will appear as shown in Fig. 2; the ground line appearing as the point O, while each plane, being

seen edgewise, will appear as a mere line. Thus the two planes form four equal dihedral angles, which are numbered **1, 2, 3, 4,** in the order shown; that in which the eye is placed, as above set forth, being the *first angle.*

Thus we have:

1st Angle	Above **H** and in front of **V**
2d Angle........	Above **H** and behind **V**.
3d Angle........	Below **H** and behind **V**.
4th Angle........	Below **H** and in front of **V**.

8. In Figs. 1 and 2, P is a point in space, here taken in the first angle; the vertical line Pp is its **horizontal projecting line,** and p is its **horizontal projection.** The line Pp' perpendicular to **V** is its **vertical projecting line,** and p' is its **vertical projection.**

This illustrates the notation adopted, the capital letter denoting a point in space, and the small letters denoting its projections, that upon the vertical plane being accented: thus we write, for example, "the point M," indicating the point whose horizontal projection is m, and whose vertical projection is m'.

These two projections suffice to determine the position of the point in space; for if in Fig. 1 we suppose p and p' only to be given, a vertical line through p and a perpendicular to **V** through p' will intersect in P.

9. In Fig. 1, draw through p a perpendicular to **V**, cutting **AB** in c; then pc is parallel and equal to Pp'. Completing the rectangle, $p'c$ is parallel and equal to Pp. That is to say, the distance of a point in space from the **vertical plane** is equal to the distance of its **horizontal projection** from the ground line; and its distance from the **horizontal plane** is equal to the distance of its **vertical projection** from the ground line.

10. Hold the page in a vertical position, and looking perpendicularly against it at Fig. 3, suppose the paper to be the plane V of Fig. 1; the line **AB** will then represent the horizontal plane seen edgewise, and the line $p'c$ will be seen in its true length and position. Next, hold the page in a horizontal position, and looking vertically downward at the same figure, imagine the paper to represent the plane H of Fig. 1. The line **AB** will then represent

the *vertical* plane seen edgewise, and the line pc will be seen in its true length and position. Thus the single plane of the paper represents both **H** and **V**, and with a little effort can at pleasure be regarded as either the one or the other. Fig. 3, then, is a representation of a point P, situated in the first angle, in **orthographic projection** upon the two **principal planes**. In reality, only the ground line and the two points p and p' are absolutely necessary; but we observe, that since in Fig. 1 pc and $p'c$ are both perpendicular to AB at the same point c, they will in Fig. 3 coincide in one right line. That is to say, **the two projections of a given point must lie on the same perpendicular to the ground line.**

11. The Profile Plane. In effect, Fig. 3 is both a *front* view and a *top* view of the model shown in Fig. 1; the space above **AB** representing the upper part of **V** and also that part of **H** which is behind **V**, while the space below **AB** represents the lower part of **V** and the front part of **H**.

Now, Fig. 2 is an *end* view of the same model, the eye looking (**7**) in the direction AB; and AB is seen in Fig. 1 to be perpendicular to the plane of the rectangle Pc: in other words, Fig. 2 is an orthographic projection upon a plane perpendicular to the ground line.

Such a plane is called a **profile plane**; the projection upon it is called simply a **profile**, and if made, as here, separate and distinct from the projections upon the principal planes, it is often of the greatest use.

Evidently, a profile may be constructed, representing the model as seen from the right, looking in the direction BA; the first and fourth angle will in that case lie on the *left* of **V**, and the order of the numbers will be the reverse of that shown in Fig. 2.

12. Location of the Profile. If the profile is made on the first supposition, (**7**) the view being *from* the left, it should be placed **at the left** of the drawing showing the projections on the principal planes; if made on the second supposition, that the view is from the right, the profile should be placed **at the right** of that drawing. Thus, Fig. 5 shows the projections on **H** and **V** of four points, one in each dihedral angle; Fig. 4 is a profile showing the same points, with their projecting lines, seen from the left; and Fig. 6

is a profile, in which the eye is at the right, looking in the direction BA.

13. If in Fig. 4 we suppose the point C, for instance, to approach the vertical plane, Cc' will be diminished, and c will approach O; when the point reaches **V**, Cc will coincide with $c'O$. Reasoning similarly with regard to **H**, we perceive that **if a point lies in either plane, it will be its own projection on that plane, and its projection on the other plane will lie in the ground line.** If it lies in both planes, the point itself and both its projections coincide in one point on the ground line.

In Fig. 5 it is observed that both projections of D, a point in the second angle, lie above **AB**, while both those of F, which is in the fourth angle, lie below **AB.**

Now, were D equally distant from **H** and **V**, its projections d and d' would fall together in one point; but the two letters would still be used. The same would in like case be true of the projections of F; and conversely, if the two projections coincide, but do not lie on the ground line, the point itself is equidistant from **H** and **V**, and therefore lies in a plane bisecting the second and fourth angles.

REPRESENTATION OF THE RIGHT LINE.

14. The projection of any line upon any plane is determined by projecting all its points upon that plane. In the case of a right line, two points determine it in space, and the projections of these two are sufficient.

Thus in Fig. 7, m and n are the projections of M and N upon the plane XY. The two projecting perpendiculars determine a plane containing the line MN; this is called the **projecting plane,** and it cuts xy in a right line mn, called its **trace**, which is the projection of the given line.

If the line is not parallel to the plane, it must pierce it when prolonged. The point of penetration must lie on the given line, and also in its projection; it is therefore at their intersection P. This point is sometimes called the **trace** of the line on the plane.

Draw, in the projecting plane Mn, a line NI perpendicular to Mm; it is also perpendicular to Nn, consequently In is a rectangle

Now regarding IN as the given line, mn is its projection; therefore, **if a line be parallel to a plane, its projection on that plane will be parallel and equal to the line itself.**

Regarding MI as the given line, the projections of all its points fall together at m; that is to say, **if a line be perpendicular to a plane, its projection on that plane is a point**; which point lies in the line itself.

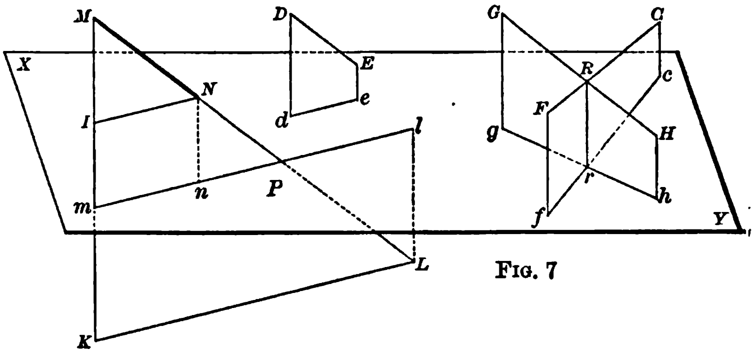


FIG. 7

If a line be inclined to a plane, its projection on that plane will be shorter than the line itself; thus, MN, being the hypothenuse of the right-angled triangle MIN, is greater than the base IN, or its equal mn.

15. If two lines are parallel in space, their projections on the same plane will also be parallel. In Fig. 7 let DE be parallel to MN; then its projecting plane De is parallel to the projecting plane Mn. The plane XY cuts these two parallel planes in the lines mn, de, which are therefore parallel to each other.

Prolong MN to any point L on the opposite side of XY, of which l is the projection; also produce Mm and draw LK perpendicular to it. Then $mK = lL$, and $mI = nN$, whence $MI = Mm - Nn$, and $MK = Mm + Ll$. Now MN is the hypothenuse of the triangle MIN, whose base IN is equal to mn, and ML is the hypothenuse of the triangle MKL, whose base is equal to ml. We see, then, that **the true length of a line in space** is equal to the hypothenuse of a right-angled triangle, whose base is equal to **the projection of the line** on any plane; the altitude being equal to the **difference of the projecting perpendiculars** of the two extremities of

the line if they lie on the **same side of the plane,** and equal to their **sum if they lie on opposite sides.**

16. If two lines intersect in space, their projections upon any plane will either intersect each other, or they will coincide. If the plane of the given lines is perpendicular to the given plane, it will be their common projecting plane, and the projections will coincide; thus *mn* is the projection of both *MN* and *IN*. But the lines *FC*, *GH*, which intersect at *R*, have not a common projecting plane; and the two planes *Gh*, *Fc*, intersect in a line *Rr*, which must be perpendicular to *XY*, and is therefore the projecting line of *R*; that is to say, the **intersection of the projections** of two intersecting lines upon the same plane **is the projection of the intersection.**

But the projections may intersect although the lines themselves do not; it is evident that in the plane *Hg* many lines may be drawn which would pass either under or over *FC*, and in the plane *Cf* many others which would not intersect *GH*.

17. A line, like a point, is represented by its projections on **H** and **V**; thus in Fig. 8, *cd* is the horizontal, and *c′d′* is the verti-

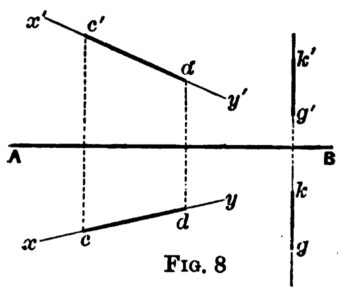


FIG. 8

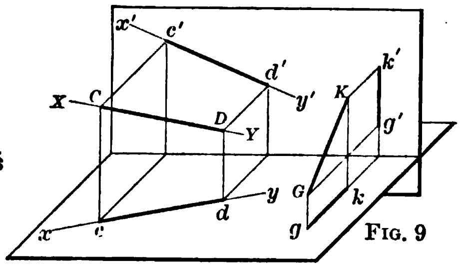


FIG. 9

cal, projection of the line *CD*. This is shown pictorially in Fig. 9, where *Cd* is the **horizontal projecting plane,** and *Cd′* is the **vertical projecting plane,** of the given line. In general, a line is fully determined by its projections; for if they are given the projecting planes can be constructed, and since the line must lie in each of them, it will be their intersection, if they have one. The limited line *CD* is a portion of the line *XY* of indefinite length, represented by indefinitely extending the projections, as *xy*, *x′y′*.

18. In general, then, any indefinite line *xy* in **H** may be assumed

as the horizontal projection, and any indefinite line $x'y'$ in **V** as the vertical projection, of a line whose position in space is thus definitely fixed.

This, however, is subject to the restriction that if either projection be perpendicular to **AB**, the other projection, whether it be a point or a right line, must lie in the prolongation of that perpendicular.

Thus in Fig. 8, the horizontal projection kg is perpendicular to **AB**; but (**10**) the vertical and the horizontal projections of any point must both lie in the same perpendicular to the ground line, consequently $k'g'$ will lie in kg produced.

In this case the indefinite projections on **H** and **V** do not suffice to determine the line; the projecting planes coincide, being perpendicular to **AB** at the same point, and have no line of intersection. If, as in Fig. 8, the projections of two points of the line are distinguished by letters, the line is determined, but such a representation is most unsatisfactory and difficult to read: the line lies in a profile plane, and its projection thereon should always be added.

19. A line in space may lie in either plane of projection; its projection on the other then falls in the ground line. If it lies in neither plane, it may be parallel to one only, parallel to both, or inclined to both.

If a line is perpendicular to one plane, it is parallel to the other; its projection on the first is a point, and its projection on the other is perpendicular to **AB**.

If the line is parallel to one plane and inclined to the other, its projection on the first is parallel to the line itself, and its projection on the other is parallel to **AB**.

If the line is parallel to both planes, the line itself and both its projections are parallel to **AB**.

20. The *doubly oblique* positions may be divided into two groups; one including those which *ascend* as they recede, like the ones drawn on the sloping plane ML, Fig. 10; and the other including those which *descend* as they recede, like the ones on the farther plane MN.

Those of the first group may cross the first angle, piercing the front part of **H** and the upper part of **V**; they may cross the third

angle, piercing the lower part of **V** and the rear part of **H**; or they may intersect **AB**, in which case they lie wholly in the second and fourth angles.

Those of the second group may cross the second angle, piercing

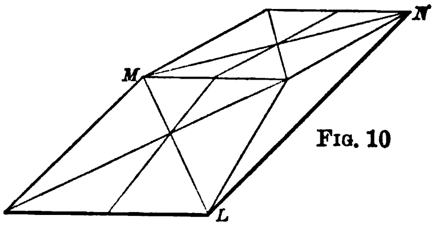


FIG. 10

the upper part of **V** and the rear part of **H**; they may cross the fourth angle, piercing the front part of **H** and the lower part of **V**; or they may intersect **AB**, in which case they lie wholly in the first and third angles.

Lines of either group, as shown in Fig. 10, may incline either to the right or to the left, in which case their projections will be inclined to **AB**; or they may do neither; lying then in profile planes, their projections on **H** and **V** are perpendicular to **AB**.

21. In Fig. 11 the projections on **H** and **V** consist merely of

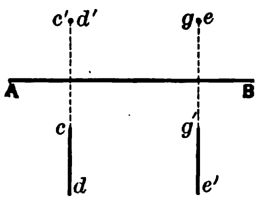


FIG. 11

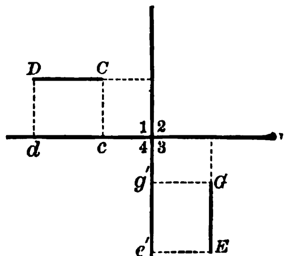


FIG. 12

two lines below **AB** and perpendicular to it, and two points above **AB**. These representations being identical, it would be impossible to distinguish between them, or to decide with certainty what either was intended to show, were they not lettered. By the aid of the letters we perceive that *CD* is a line of limited length, perpendicular to **V**, and lying in the first angle, while *GE* is a limited vertical line situated in the fourth angle. These things are seen by a single

glance at Fig. 12; which illustrates the value of the profile, sometimes in even very simple cases.

22. In Fig. 13 are given the projections of a horizontal line lying, as pictorially represented in Fig. 14, in the second angle. Being horizontal, its vertical projecting plane is also horizontal, and therefore cut by **V** in a horizontal line; that is to say, the vertical projection $c'd'$ is parallel to **AB** (**19**).

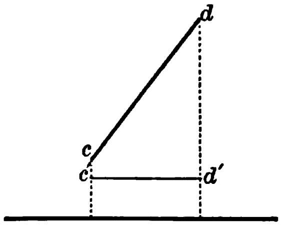


FIG. 13

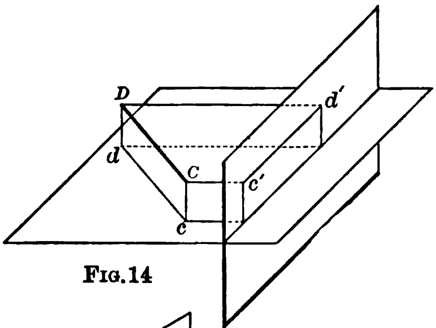
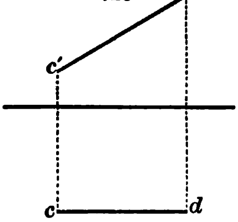


FIG. 14

FIG. 15



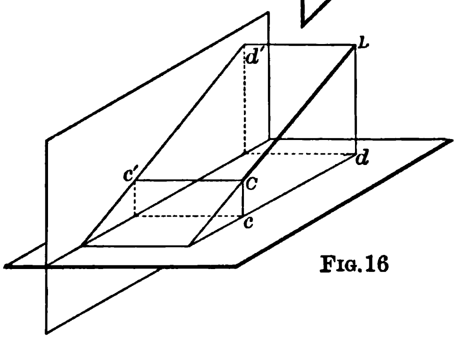


FIG. 16

Fig. 15 gives the projections of an inclined line, in the first angle and parallel to **V**, as seen in Fig. 16. Its vertical projecting plane is therefore parallel to **V**, and since these two parallel planes are cut by **H** in parallel lines, the horizontal projection cd is parallel to **AB** (**19**).

23. The doubly oblique lines, not being parallel to either plane, pierce them both. In relation to these, beginners sometimes find difficulty in *reading* the drawings,—that is, in forming by aid of the projections alone, clear perceptions of the actual positions of the lines in space. This difficulty may perhaps be lessened by consider-

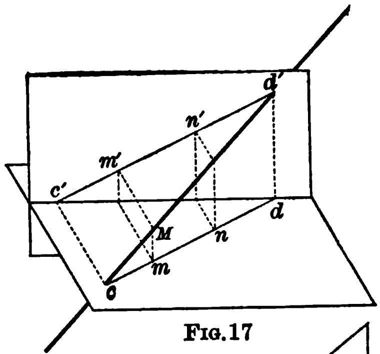


FIG. 17

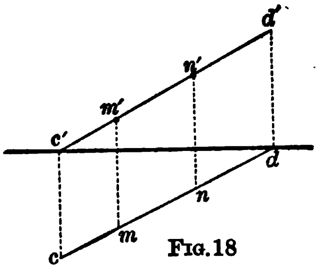


FIG. 18

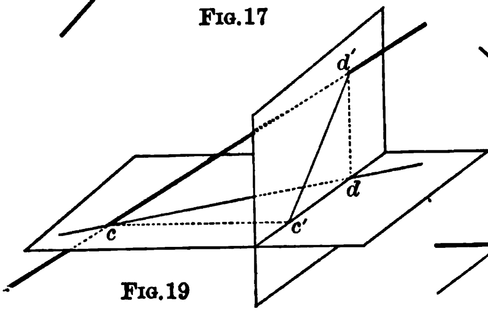


FIG. 19

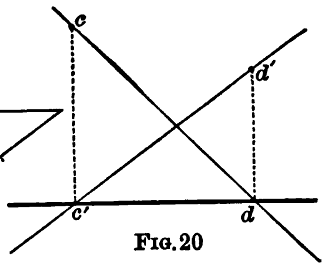


FIG. 20

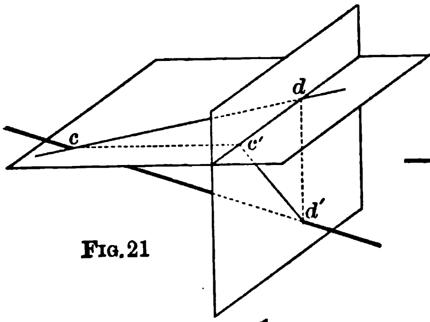


FIG. 21

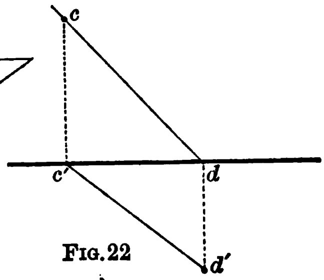


FIG. 22

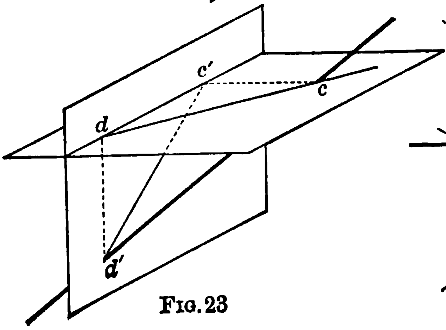


FIG. 23

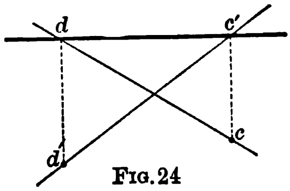


FIG. 24

ing at first such as do not meet the ground line, and confining the attention to the portions intercepted between **H** and **V**.

In Fig. 17 is given a pictorial representation of a line which crosses the first angle. In order to draw the projections of such a line we may assume c, Fig. 18, as the horizontal projection of any point lying in **H** in front of **V**; its vertical projection c' will lie in **AB**: also assume d' as the vertical projection of a point in **V** above **H**; its horizontal projection d must also lie in **AB**; therefore cd is the horizontal and $c'd'$ is the vertical projection of the required line.

24. So, by assuming the projections of a point in each plane, it is easy to represent a line crossing any angle at pleasure. Thus Fig. 19 shows one which crosses the second angle; its representation in projection, Fig. 20, differs from Fig. 18 only in this, that c lies above instead of below **AB**, being the horizontal projection of a point in **H** behind **V**.

Completing the series, the four following figures represent, pictorially and in projection, lines crossing the third and the fourth angles. And it is observed that the two projections of the intercepts in the **first** angle, Fig. 18, and in the **third** angle, Fig. 22, **do not intersect each other.** If the intercept lies in either of the other angles, **its projections cross each other**, the intersection being *above* **AB** if it lies in the **second**, as in Fig. 20, and *below* **AB**, as in Fig. 24, if it lies in the **fourth**.

25. To find the traces of a line whose projections are given. The points in which a line pierces **H** and **V** are called respectively its **horizontal trace** and its **vertical trace.** In constructing Fig. 18, as explained in (**23**), the traces were assumed, and determined the projections of the line; by merely reversing the process, the traces may be found if the projections are given. For if the line in space be produced, its projections will be extended; the distance of any point of the horizontal projection from **AB** shows the distance of the corresponding point in the line from **V**, which becomes zero when that projection meets **AB**; and similarly the altitude, or distance from **H**, becomes zero when the vertical projection meets **AB**. Therefore, if in Fig. 18 the projections mn, $m'n'$ are given, produce mn to cut **AB** in d; this will be the horizontal projection of the vertical trace: the other projection must lie on a perpendicu-

lar to **AB** at d, and also on the prolongation of the vertical projection $m'n'$; therefore it is at their intersection d'. Produce $m'n'$ to cut **AB** at c', the vertical projection of the horizontal trace; at c' draw a perpendicular to **AB**, cutting mn produced in c, the other projection. The points C and D are the traces sought.

26. In attempting to find the traces of a line given in this manner, it may happen that both its projections meet the ground line at the same point. This means simply that the line pierces both **H** and **V** at that point, and therefore intersects the ground line. This is the case in Fig. 25; and as the line MN lies in the first angle, it must when prolonged pass into the third angle after crossing **AB**, as more distinctly seen in the profile, Fig. 26, the addition of which third

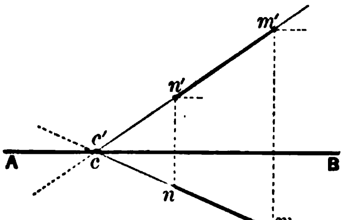


FIG. 25

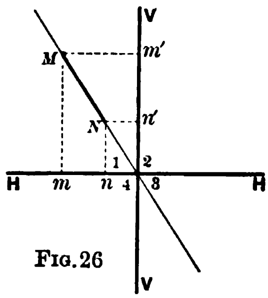


FIG. 26

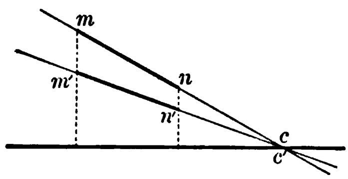


FIG. 27

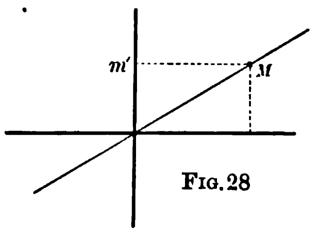


FIG. 28

projection greatly facilitates the reading in such cases. In Fig. 27 the given limited portion MN of the line lies in the second angle; when prolonged the line cuts **AB** at C, and, as shown in the profile, Fig. 28, passes on into the fourth angle.

27. Oblique Lines with Coincident Projections. In Fig. 29 the point M is as far behind **V** as it is above **H**, and in consequence m and m' fall together. The point D is as far below **H** as it is in front of **V**, whence d and d' also fall together. Therefore the vertical and the horizontal projections of the line MD coincide in one line.

Every other point of the line will therefore be represented by coincident projections, as n, n'; all such points lie in either the second angle or the fourth, and since they are equidistant from **H** and **V**, they and the line itself lie, as shown in the profile, Fig. 30, in a plane which bisects those two angles (**13**).

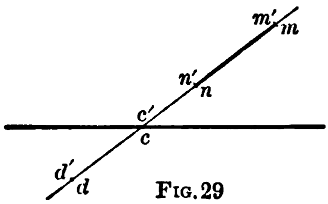


FIG. 29

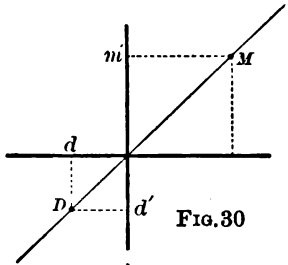


FIG. 30

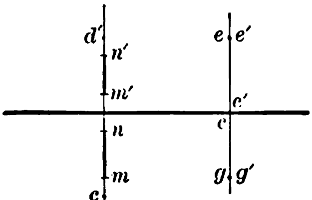


FIG. 31

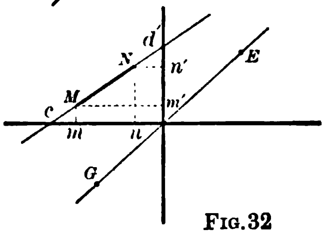


FIG. 32

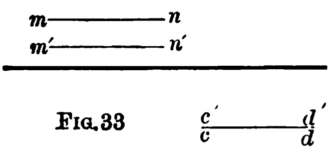


FIG. 33

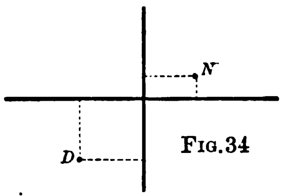


FIG. 34

28. Oblique Lines in Profile Planes. In Fig. 31 both projections are perpendicular to the ground line. In this case these projections upon the principal planes are utterly inadequate to convey a clear idea of the position of a limited portion of the line, even with the aid of the letters. The *indefinite* projections, even supposing it to be known which is the vertical and which the horizontal, do not suffice (**18**) to locate the line in space, and since their prolongations coincide, it is impossible by their use to find the traces. The obvious and the only sensible course is to make an **independent and detached drawing** in profile, as shown in Fig. 32, which exhibits in the clearest possible manner the position of the line in relation to both **H**

and **V**, whether it crosses one angle or another, like *MN*, or like *GE* intersects the ground line.

29. Lines Parallel to Both Principal Planes. A line which is parallel to both **H** and **V** is parallel to the ground line. It may lie in one of those planes; where it is its own projection, the projection on the other plane falling in **AB**. If it does not lie in either **H** or **V**, both its projections are parallel to the ground line. The projections of such a line upon the principal planes are sufficient to locate the line in space, and, as shown in Fig. 33, they suffice to represent it. Nevertheless, in this case also the reading of the drawing is facilitated, and the location of the line more clearly indicated, as shown in Fig. 34, by adding a profile.

REPRESENTATION OF THE PLANE.

30. The intersection of a plane with **V** is called its **vertical trace**; its intersection with **H** is called its **horizontal trace**; and the plane is represented by drawing these traces.

Any horizontal plane, being parallel to **H**, has no horizontal trace, and its vertical trace is parallel to **AB**. Example: the vertical projecting plane of *CD*, Fig. 14.

If a plane is parallel to **V** it has no vertical trace, and its horizontal trace is parallel to **AB**. Example: the horizontal projecting plane of *CD*, Fig. 16.

If a plane is parallel to **AB** and inclined to **H** and **V**, both traces will be parallel to **AB**; or, they may coincide in the ground line itself. These cases will be illustrated farther on.

If a plane is perpendicular to **AB**, i.e., if it is a profile plane, both traces are perpendicular to **AB**. Example: the projecting planes of *KG*, Fig. 9.

It will be perceived from the examples here quoted that the projections of all lines are the traces of their projecting planes (**14**).

31. If a plane be inclined to **AB**, it will cut it in a point; the traces must intersect at this point, and one or both of them will be inclined to **AB**. Thus in Fig. 35, the oblique plane *MN* cuts **AB** at *D*; its horizontal trace is *dDc*, and *d'Dc'* is its vertical trace.

Such a plane is represented in projection as in Fig. 36; it is designated and referred to as the plane *dDd'*. If, as often hap-

pens, attention is to be confined to that portion of the plane which lies in the first angle, between **H** and **V**, the parts Dc, Dc' of the traces are omitted; but it must be kept in mind that the plane is capable of indefinite extension, and both traces can be indefinitely produced.

It is to be observed, in regard to this notation, that d and d' are not used to indicate the two projections of the same point; d merely designates a point in **H**, and d' designates a point in **V**. If the location in space of a particular point in either trace is to be indicated,

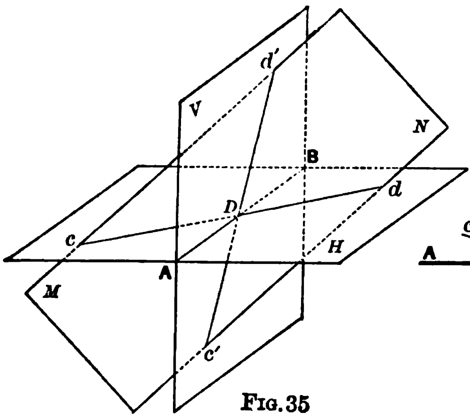


FIG. 35

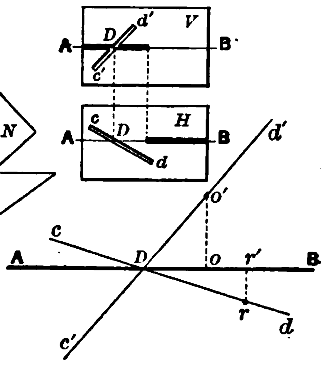


FIG. 36

the two projections of that point are lettered in the usual manner; thus o, o' are the projections of a point in Dd', and r, r' those of one in Dd; these points are referred to as O and R respectively.

32. For the purpose of aiding those who may at first find difficulty in reading the diagram, Fig. 36, there are placed above it drawings, on a reduced scale, of the two cards with their slots which form the planes of projection in the model, Fig. 35.

In looking at the card V, the paper is held, or supposed to be held, in a vertical position; while in looking at the card H, it is held, or imagined to be, in a horizontal position, and viewed from above. The diagram represents both these cards in skeleton; make therefore the same suppositions in regard to the position of the paper and the direction in which it is to be viewed, closing the mental eye to one projection while studying the other. By persistent efforts of this kind, the power may gradually be acquired of

reading the diagrams with ease—that is to say, of forming by the aid of the projections alone, clear mental images of the positions and relations of the lines and planes which they represent, so that they will, as one may say, stand out in relief with stereoscopic distinctness.

33. In regard to these two cards, it is evident that the directions of the slots are entirely arbitrary and independent of each other; but when put together, the point *D* on one must coincide with the point *D* on the other. Which is only another way of saying that from the same point on **AB** we may draw one line in any direction on **H**, and another in any direction on **V**, and these two lines will determine a plane, of which they are the traces.

If the vertical trace is perpendicular to **AB**, the plane is vertical, but may make any angle with **V**, as in the swinging of a common door upon its hinges. If the horizontal trace is perpendicular to **AB**, the plane is perpendicular to **V**, but may make any angle with **H**; as in the opening of a trap-door whose hinges are perpendicular to the wall. The plane *dDd'* illustrates the former, and the plane *tTt'* illustrates the latter, of these two cases, in Figs. 37 and 38.

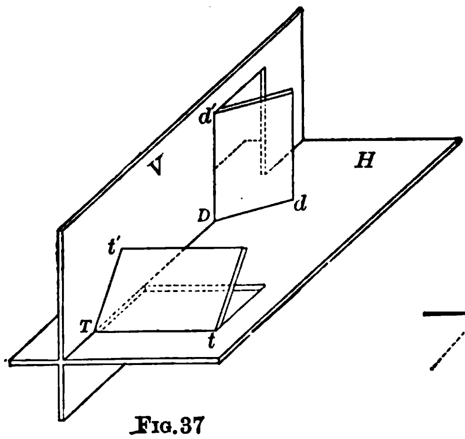


FIG. 37

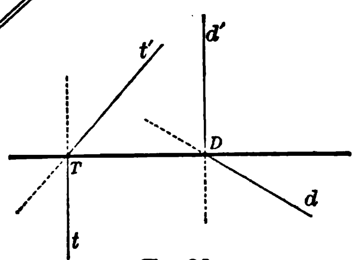


FIG. 38

34. In relation to the angle included between the parts of the traces in front of **V** and above **H**, it is apparent that the angle pictorially represented by *dDd'* in Fig. 35 is in fact acute, while in Fig. 37 *dDd'* and *tTt'* represent right angles. In Fig. 39 *dDd'*

represents an obtuse angle; and in the diagram, Fig. 40, are given the traces of the same plane, similarly lettered. Above are added the small drawings of the cards with their slots, for forming the planes **H** and **V** of the model.

These slots are inclined to *AB* in the same direction, though the angles are somewhat different in the two cards. A moment's reflection will show that these angles might be made exactly the same, and also that if they were, the two traces *dD*, *Dd'*, instead of forming an angle with each other in the diagram, would form

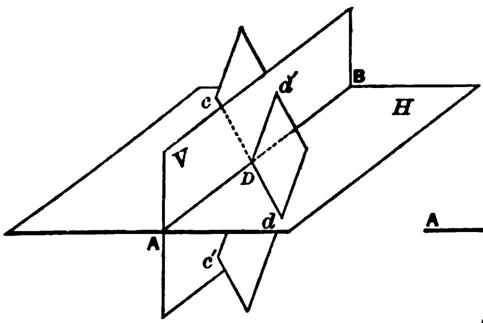


FIG. 39

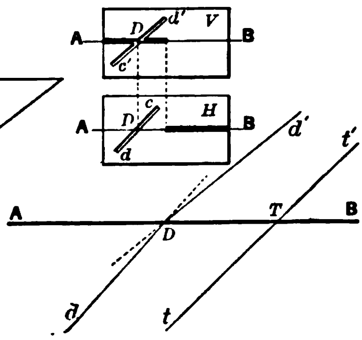


FIG. 40

one continuous right line; which is the case with the traces of the plane *tTt'*. Clearly, the position of the plane shown in Fig. 39 would be but slightly changed by this modification.

35. In Fig. 41, the plane *TT* is parallel to **AB**; consequently its horizontal trace *tt* and its vertical trace *t't'* are both parallel to the ground line. The plane *DD* passes through **AB**, which therefore constitutes both traces.

The diagram representing these two planes is given in Fig. 42; but it is very obvious that they are represented much more clearly in the profile, Fig. 43.

Since any number of planes may be passed through the ground line, the position of any one of them must be determined by some other condition; but when it is determined, its true relation to **H** and **V** is at once shown by the detached profile.

In Fig. 44, the horizontal trace *dd* coincides with the vertical trace *d'd'*, the former being as far behind **V** as the latter is above

H. Again, *tt* is the horizontal trace of a plane, and lies as far in front of **V** as the vertical trace *t′t′* is below **H**, so that these two traces are also represented by one line. Finally, *mn*, *m′n′* are the projections of a line, parallel to **AB**, in the fourth angle, and equidistant from the principal planes. The superiority of the profile,

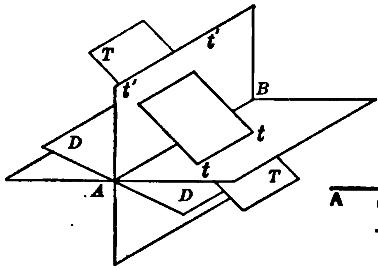


Fig. 41

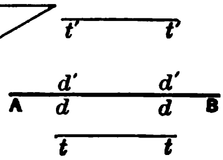


Fig. 42

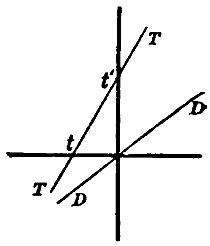


Fig. 43

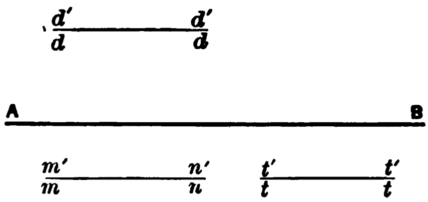


Fig. 44

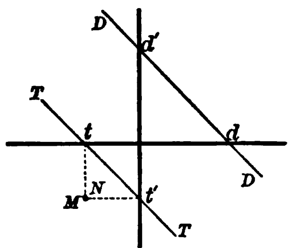


Fig. 45

Fig. 45, in respect to distinctness and ease of comprehension, is too obvious to require comment.

GEOMETRICAL PRINCIPLES AND DEDUCTIONS.

36. If two planes intersect, any line of either will pierce the other in a point of their common line, if at all; hence—

1. If a line lie in a plane, the traces of the line will be points in the corresponding traces of the plane.

2. To draw a line in a given plane, join any point in one trace with any point in the other.

3. To draw a plane containing a given line, join the traces of the line with any point on **AB**.

Any horizontal line in a given plane is parallel to the horizontal trace, pierces **V** in a point of the vertical trace, and its vertical projection is parallel to **AB**.

If a line in a given plane is parallel to **V**, it is parallel to the vertical trace, pierces **H** in a point of the horizontal trace, and its horizontal projection is parallel to **AB**.

If a plane contain any two lines, it will also contain any third line which cuts those two.

37. In illustration of the above: Let it be required to draw a line in the plane tTt', Fig. 46. Assume c as the horizontal projection of a point in the horizontal trace; its vertical projection is c' in the ground line. Let d' be the vertical projection of a point in the other trace, then its horizontal projection is d in the ground line; cd, $c'd'$, are the projections of a line which lies in the given plane. It follows from this, that if one projection of a point in a given plane be assumed, the other can be found by drawing through the assumed one, the corresponding projection of a line in the plane. Then the other projection of the point must lie on the other projection of the line. For example, suppose the horizontal projection o in Fig. 46 to have been assumed. Join o with any point c of the horizontal trace, and produce this horizontal projection to cut **AB**

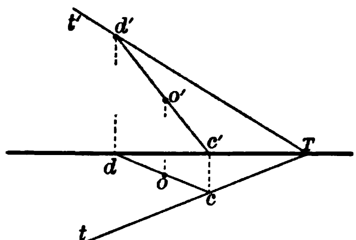


FIG. 46

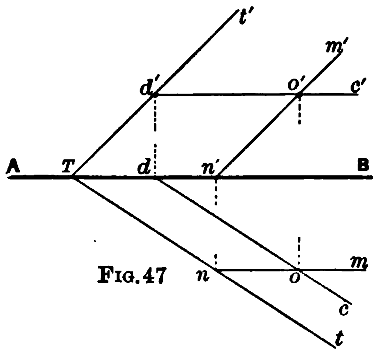


FIG. 47

in d. Since this line is to lie in the plane, its vertical projection is $c'd'$, upon which must lie the vertical projection o'. The point O thus determined lies in the given plane.

Again, let it be required to draw in the plane tTt', Fig. 47, a horizontal line at a given distance above **H**.

Draw $c'd'$ parallel to **AB**, at the given distance above it: this is the vertical projection of the required line, and d' that of its vertical trace, which is horizontally projected at d in **AB**. Therefore dc parallel to Tt is the horizontal projection of the required line.

Let it be further required to draw in the same plane a line parallel to **V**, at a given distance in front of it. The horizontal projection is mn, parallel to **AB** and at the given distance below it; the line pierces **H** at the point N, whose vertical projection is n' on **AB**, and $n'm'$ parallel to Tt' is the vertical projection of the required line.

38. The two lines CD and MN evidently intersect; and since they cannot intersect in more than one point, the test of the accuracy of the constructions lies in this, that the intersection o of the horizontal projections, and the intersection o' of the vertical projections, lie on the same perpendicular to the ground line.

39. To draw two lines which shall intersect. This may be done by assuming the point of intersection C, Fig. 48; c and c' must necessarily lie on the same perpendicular to **AB** (**10**). The horizontal projection of each line must pass through c, and its vertical projection through c'; but the directions of mcn, $m'c'n'$, as well as those of pcr, $p'c'r'$, are entirely arbitrary, with the exception that if one projection of either line is perpendicular to **AB**, the other projection of that line must be so likewise (**18**).

The two lines GL, DE, in Fig. 48, intersect at O; the horizontal projections intersect at o, but the vertical projections coincide. This merely shows that the plane determined by the two

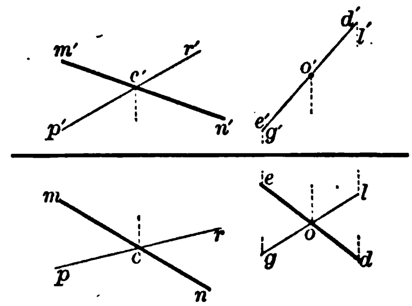


FIG. 48

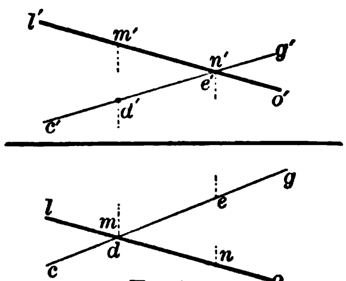


FIG. 49

lines is perpendicular to **V**, and is their common vertical projecting plane (**16**).

40. To draw two lines which shall not lie in the same plane. Neither the vertical nor the horizontal projections can coincide, since if they did the two lines would have a common projecting

plane. The horizontal projections must therefore cross each other, and so must the vertical projections; but these two points of intersection **must not lie in the same perpendicular to AB.**

Thus in Fig. 49, the horizontal projections of the lines CG, LO, intersect; let this point of intersection be the horizontal projection m of a point M upon LO, then its vertical projection is m' upon $l'o'$. Let the same intersection be the horizontal projection d of a point upon CG, then its vertical projection is d' upon $c'g'$. Similarly, the intersection of $l'o'$ and $c'g'$ is the common vertical projection of two points, N upon LO, and E upon CG.

41. If two parallel planes are cut by a third plane, the lines of intersection are parallel. Therefore, in order to represent a plane parallel to one of which the traces are given, draw the vertical trace of the second parallel to that of the first, and the horizontal trace of the second parallel to the horizontal trace of the first. If the new plane is required to be so located as to satisfy some special condition, it is clear that the determination of one point in either trace is sufficient.

For example: Let it be required to draw a plane parallel to the given plane tTt', Fig. 50, through the given point O. Draw, through the given point, a line parallel to the horizontal trace; its

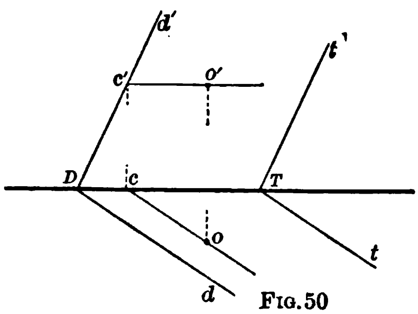


FIG. 50

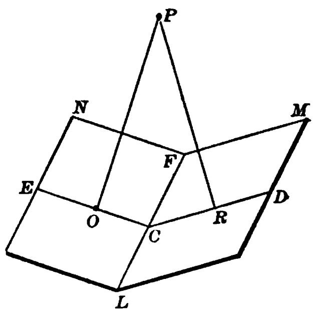


FIG. 51

horizontal projection passes through o and is parallel to Tt, its vertical projection passes through o' and is parallel to **AB**. This is a line of the required plane, and pierces **V** at the point C, whose vertical projection c' is a point in the vertical trace, which is $d'c'D$ parallel to Tt'. This trace cuts **AB** at D, and the horizontal trace Dd is parallel to Tt.

42. Let LM, LN, Fig. 51, be two planes, and LF their line of intersection. From any point P let fall upon these planes the perpendiculars PR, PO; these two lines determine a plane which is perpendicular to both the others, and therefore to LF; it also cuts them in the lines CRD, COE, which meet at C on LF. But LF is perpendicular to the plane OPR, and therefore to the lines CE, CD, which pass through its foot in that plane. Now, regarding LM as a plane of projection, and LF as the trace upon it of any plane LN; then CR is the projection of PO, a perpendicular to LN, and the trace LF is perpendicular to the projection CR.

43. Therefore, **if a line be perpendicular to a plane,** the **vertical** projection of the line will be **perpendicular** to the **vertical** trace of the plane, and the **horizontal projection** will be perpendicular to the **horizontal trace.**

And conversely: if the projections of the line are **respectively** perpendicular to the traces of the plane, the line itself is perpendicular to the plane.

In illustration, let it be required to draw through the point O, Fig. 52, a line perpendicular to the plane tTt. Since the projec-

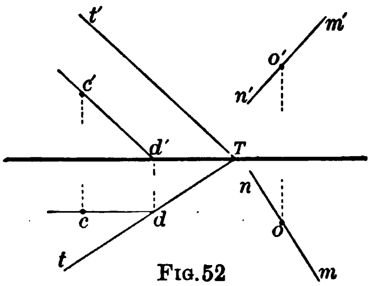


FIG. 52

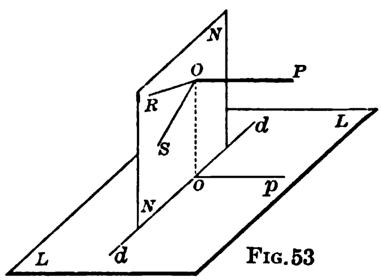


FIG. 53

tions of the line must pass through those of the point, we have merely to draw $m'o'n'$ perpendicular to Tt', and mon perpendicular to Tt; then MN is the required line.

Or, having given the line MN and the point C, let it be required to draw a plane through that point and perpendicular to the line. Here again, since the *directions* of the traces are known, the determination of one point in either trace suffices to locate the plane; and this is effected by drawing through the point a parallel to the other trace. Thus, to find a point in the horizontal trace, draw

through C a line parallel to the vertical trace; its vertical projection is $c'd'$ perpendicular to $m'n'$, its horizontal projection is cd parallel to **AB**, and it pierces H in the point d, d'. The horizontal trace of the required plane is therefore tdT perpendicular to mn, and the vertical trace is Tt' perpendicular to $m'n'$.

44. In Fig. 53, let LL be a plane of projection, NN a projecting plane perpendicular to it, of which dd is the trace, and PO a line perpendicular to NN and consequently parallel to LL. Then po, the projection of PO, is parallel to that line itself, and therefore perpendicular to NN and to its trace dd. Now PO, being perpendicular to NN, is perpendicular to all right lines drawn through its foot in that plane, as OR, OS; and the projections of all these lines fall in the trace dd, which is perpendicular to po; consequently we have, that **the projection of a right angle will be a right angle, if one of its sides is parallel to the plane of projection.**

REVOLUTION AND COUNTER-REVOLUTION.

45. The axis of a circle is a right line passing through its centre, and perpendicular to its plane. **A point is said to revolve about a right** line as an axis, when it describes the circumference of a circle **whose centre is in the axis, and whose plane is perpendicular to the axis.**

When all the points of any geometrical magnitude move in this manner, without change of relative position and therefore with the same angular velocity, the whole magnitude is said to revolve about the right line as an axis.

It will be found in subsequent operations, that magnitudes under consideration can often be thus revolved into positions in which certain processes can be more conveniently executed; after which they are revolved back again into their original positions; and this restoration is called **counter-revolution.**

46. One of the simplest examples of this kind of manipulation is the following: *Given*, a line lying in a plane of projection, and a point not in the plane; *Required*, to show where the point will fall, when revolved about the line into the plane.

This is pictorially illustrated in Fig. 54, where EF is the line, lying in the plane LL, and P the given point. The plane of rota-

tion is NN, perpendicular to EF and therefore to LL; it contains the projecting line Pp of the given point, and p lies in the trace dd, which is perpendicular to EF and cuts it at C, the centre of the circular path of P.

Consequently the point will fall in the plane LL either at r or

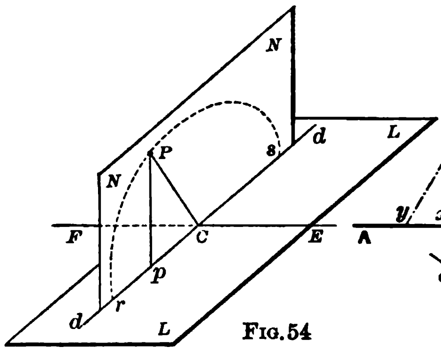


FIG. 54

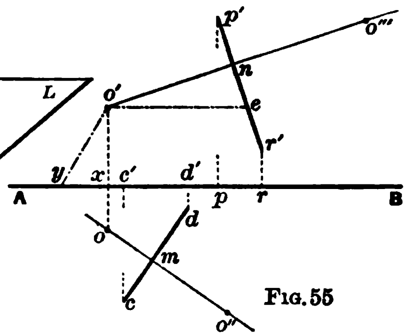


FIG. 55

at s on the trace dd, at a distance from C equal to PC the radius of the circle. And it is seen that PC is the hypothenuse of a triangle, of which the altitude Pp is **the distance of the point from the plane**, and the base pC is **the distance of the projection of the point from the axis.** The axis may pass through this projection; that is, p may coincide with C, in in which event the projecting line Pp is the radius of the circle.

47. The construction in projection is shown in Fig. 55, where O is the given point. *First:* to revolve this point about CD into the horizontal plane. Draw through o, the horizontal projection of the point, an indefinite perpendicular to cd, cutting it in m; the point will fall somewhere upon this line, which is the trace of the plane of rotation. The distance of the point from the plane is $o'x$ the projecting line, and om is the distance of the projection from the axis. Set off $xy = om$, then $o'y$ is the true distance of the point from m; therefore set off $mo'' = o'y$, and o'' is the position of O after revolution.

Second: to revolve O about PR into the vertical plane. Draw through o' an indefinite perpendicular to $p'r'$, cutting it at n, and on this perpendicular set off no''', equal to the hypothenuse of a triangle whose base is $o'n$, and whose altitude is equal to xo, the

distance of O from **V**: this triangle may be conveniently constructed by setting off on $p'r'$, $ne = xo$, giving $o'e$ as the true distance of O from n. The line $o'e$ is of course not actually to be drawn, but its length being taken in the dividers, is set off from n as no''', and o''' thus located is the required position; similarly, $o'y$ is measured, but not drawn.

48. Since a right line and a point outside of it are sufficient to determine a plane, the point O in Fig. 55 may be regarded as lying in a plane of which cd is the horizontal trace; and when this point has been revolved about that trace into the position o'', it is clear that the whole plane just mentioned will coincide with **H**, so that any lines or figures which it had contained originally, or which may now be drawn upon it, will be seen in their true forms, dimensions, and relative positions.

In like manner, the same point O lies in another plane of which $p'r'$ is the vertical trace, and when O reaches o''', that plane coincides with **V**. Any plane, then, can be revolved about either trace into the corresponding plane of projection; and it is clear that by revolving it about a line parallel to one of those traces, it may be made **parallel** to the corresponding plane.

49. Counter-revolution of a Plane. In Fig. 56, let the point P be revolved about the axis EF into the plane of projection LL as in Fig. 54, p'' being its revolved position, and C the centre of its

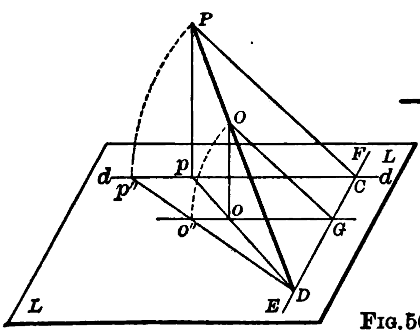


FIG. 56

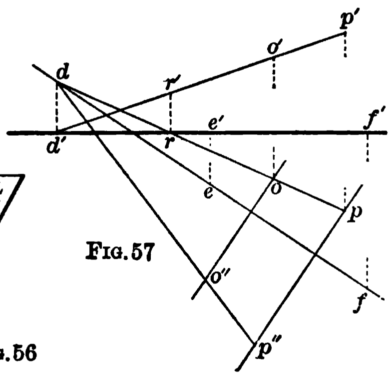


FIG. 57

circular path. The plane PCE now coincides with LL; assuming o'' to be the revolved position of any point therein other than P,

it is required to find its position when the plane is revolved back again.

Draw $p''o''$ and produce it to cut the axis in D; then the required original position of o'' must lie on the line PD, whose projection on LL is pD. The plane of rotation of the second point is parallel to that of the first, and its trace is $o''G$ perpendicular to EF; which cuts pD in o, the projection of the point sought. Therefore erect at o a perpendicular to LL, cutting PD in O, the required point.

50. This construction, in projection, is shown in Fig. 57, where EF in the horizontal plane is the axis, P the first point in its original position, and p'' its revolved position determined as in Fig. 55. Take o'' as the revolved position of the second point; draw $p''o''$ and produce it to cut the prolongation of fe in d. This is the horizontal projection of a point in **H**, therefore its vertical projection is d' in **AB**, and pd, $p'd'$, are the projections of the line PD which contains the point sought. In the counter-revolution, o'' describes a circle in a vertical plane perpendicular to the axis, of which the horizontal trace is perpendicular to ef, and cuts pd in o, the horizontal projection of the required point; o' in $p'd'$ is its vertical projection.

SEPARATE CONSTRUCTION OF THE VERTICAL AND HORIZONTAL PROJECTIONS.

51. The combination of the projections on **H** and **V** in one diagram, and the use of **AB** to represent one plane in reading the vertical and the other in reading the horizontal projection, is often a source of perplexity at first, even to those thoroughly familiar with the various views which represent solid objects in ordinary mechanical drawing. Moreover, this combination is often a cause of excessive and needless obscurity; the various lines used in representing magnitudes, operations of revolution, counter-revolution, and what not, upon one plane, becoming so interwoven with those of the projections upon the other, as to present a bewildering maze even to the expert in reading such diagrams.

Now, the fact that these two projections can be and always have been thus combined is not at all a good reason why they always

should be. They may be constructed separately as shown in Fig. 58, where $m'n'$ is the vertical projection of a line, HH representing the horizontal plane; mn is the horizontal projection, VV representing the vertical plane; and at the left of the vertical projection is the profile, where the two planes are shown in their true relative positions. These three views correspond to what in working drawings are called the front elevation or side view, the top view or plan, and the end view or end elevation. For the purpose of comparison, the combined diagram of the projections of the same line is shown in Fig. 59; and there are no doubt many to whom the latter will seem far less clear, and more difficult to read, even in regard to so simple a magnitude as this oblique line.

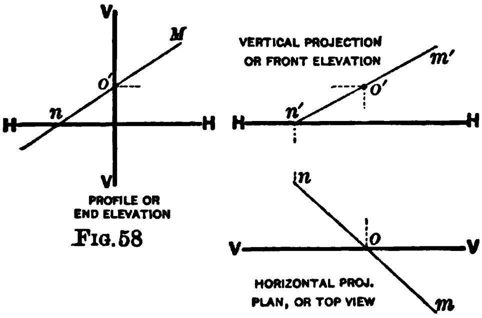


FIG. 58

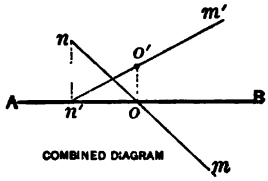


FIG. 59

In what follows, both methods will be used, as circumstances may indicate one or the other to be the more convenient. The student, of course, may use either in any case at pleasure. It is desirable that he should become familiar with both, but in any given construction he alone can tell which method seems to him the clearer, and that is the one for him to adopt.

SUPPLEMENTARY PLANES OF PROJECTION.

52. Thus far but three planes of projection have been considered, viz., **H**, **V**, and the profile plane. It will, however, frequently be found convenient to make use of others, which may be called supplementary planes; upon these the object is projected, remaining fixed in respect to the principal planes.

The positions of such supplementary planes are determined wholly by conditions of convenience, and therefore depend upon

the nature of the object; but they are in the great majority of cases such that the planes are perpendicular to one of the principal planes; indeed, it may be said that they are probably more often vertical than otherwise.

53. The use of such a plane is pictorially represented in Fig. 60, in which OR is, let us say, an obliquely placed wire, supported

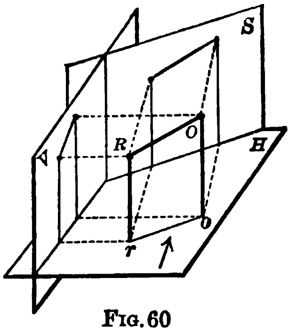


Fig. 60

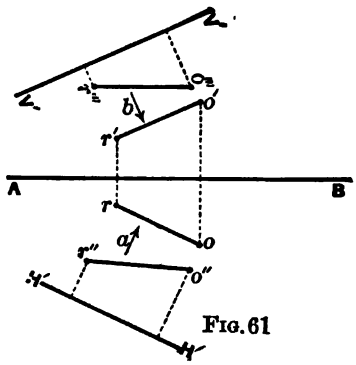


Fig. 61

by two vertical ones fixed in the horizontal plane. There are shown the projections of these not only upon **H** and **V**, but upon a supplementary plane **S**, in this case parallel to the horizontal projecting plane Or. The three lines are therefore projected upon **S** in their true lengths and relative positions, while upon **H** and **V** they are not; and it is for the purpose of thus conveying directly information which the other views do not give explicitly, that such supplementary projections are chiefly employed.

It is obvious that in viewing **S** perpendicularly, as indicated by the arrow, the axis of vision is parallel to **H**, which, being thus seen edgewise, will in projection be represented by a line bearing the same relation to this new drawing that the original ground line bears to the vertical projection.

54. This is shown in Fig. 61, where or, $o'r'$ are the projections upon **H** and **V** of a limited oblique line, like OR in Fig. 60. Below is drawn a supplementary view, looking perpendicularly against the horizontal projecting plane, as shown by the arrow a.

The horizontal plane is represented by the line $H'H'$ perpendicular to the arrow, and the points o and r, being projected perpendicularly toward this line, appear in the new view at distances

above it equal to those of o' and r' above **AB**. To use a geographical illustration, if the projection upon **V** be regarded as a view from the south, the observer looking due north, this supplementary projection is a view of the same object from the southwest, the observer looking northeast; by this change of position on the part of the spectator the altitudes of the various parts of the object are clearly neither increased nor diminished. Also, the line OR is now seen in its true length.

55. But again, the object still retaining its original position, the eye may be supposed to be above and at the same time either to the right or to the left of it, and to be directed, not vertically, but obliquely downward, yet still in a line parallel to **V**. In this case the *vertical* plane will be seen edgewise, but the vertical projecting lines will remain unchanged in length, so that all the points of the object will appear just as far from that plane as in the original horizontal projection.

In illustration, a supplementary view is given in the upper part of Fig. 61, looking in the direction shown by the arrow b. The line $V'V'$, perpendicular to the arrow, represents the vertical plane toward which the points o', r' are projected, their distances from $V'V'$ being equal to those of the horizontal projections o, r, from **AB**.

56. Such supplementary projections, like the profiles, should always be constructed as detached and independent views; their precise location is of course arbitrary, but should always be such as to prevent the possibility of confounding the lines with those of the other views.

CHAPTER II.

ELEMENTARY PROBLEMS RELATING TO THE POINT, RIGHT LINE, AND PLANE.

57. It is necessary to make a clear distinction between the **solution** of a problem and the **representation** of that solution.

The solution is effected by abstract reasoning: one link after another being added to a chain of logical arguments until a definite conclusion is reached which demonstrates that the object sought can be accomplished in a certain way. This is a purely mental process; clear conceptions can be formed in the dark, or by a blind man, of the magnitudes involved, of their relations to each other, of the various steps to be taken and their results—in short, of the complete *solution* of any problem; which is wholly independent of its *representation* and of any graphic operation whatever.

The processes of descriptive geometry, on the other hand, are purely graphic. And it is the province of this science to explain the methods, not of solving problems, but of exactly representing the data, steps, and results of solutions already effected by mathematical reasoning. This distinction is natural and inevitable, because before a thing can be represented it must be known what that thing is.

58. Analysis and **Construction.** A problem being enunciated, then, its treatment will consist of two distinct parts. *First*, a clear statement of the principles and reasoning employed in the solution and applied to the magnitudes in space; this is the **analysis.** *Second*, an explanation, in due order, of the lines employed in representing, on paper, the problem and its solution; this is called the **construction** of the problem.

59. Method of Study. The same processes may in general be applied to magnitudes under widely varying conditions; and in the

nature of things but a limited number of cases, and often only one, can be worked out in illustration. Consequently great care should be taken to avoid associating the solution of any problem with the appearance of the figure, because the assumption of different data may result in the production of a totally dissimilar diagram. This is of especial importance in regard to these elementary problems, because they are subsequently to be used as mere steps in the solution of more complex ones, and the conditions thus fixed may be quite unlike those previously met with. The best course therefore is to dispense, as soon and as far as possible, with all reference to the illustrations; first mastering the analysis and fixing in mind the successive steps, and then making an original construction by applying them to assumed data.

60. PROBLEM 1. *To find the true distance between two points given by their projections.*

Analysis. The required distance is the length of the right line joining the two points. If either projecting plane of this line be revolved about its trace into the corresponding plane of projection, the line will be seen in its true length.

Construction. This is represented pictorially in Fig. 62, and Fig. 63 shows it in projection, M and N being the given points. Revolving the horizontal projecting plane of MN about mn into **H**,

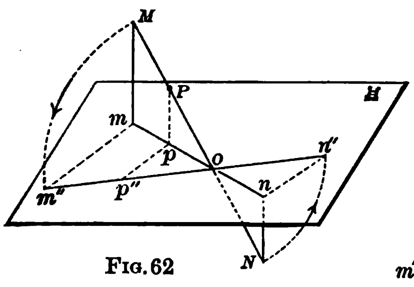


FIG. 62

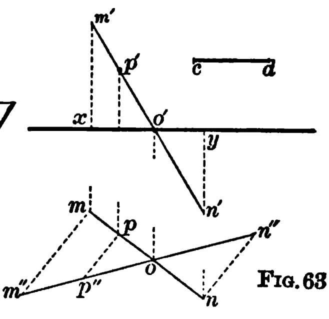


FIG. 63

the point M goes to m''; mm'' being perpendicular to mn and equal to $m'x$, the horizontal projecting line. Similarly, N goes to n'', the distance nn'' being equal to $n'y$; and $m''n''$ is the distance required.

The required distance may also be ascertained by means of a supplementary projection, as shown in Fig. 61, or by means of an

independent construction, apart from the drawing, as explained in (**15**).

N.B. In this case the distances mm'', nn'', are set off in opposite directions, because the points M and N are on opposite sides of the axis. Had they been on the same side, as M and P are, these distances would have been set off in the same direction. In either case, the line joining the two points, if not parallel to the plane into which it is revolved, will pierce it if prolonged, as at o, o' in Fig. 63. This point of penetration, being in the axis, remains fixed, and the given line must pass through it in its revolved as well as in its original position; thus $m''n''$ passes through o.

61. If it is required to set off from M a distance along MN equal to a given line cd, first revolve the line MN into **H** as above, and then lay off $m''p''$ equal to cd. In the counter-revolution, p'' goes, in a direction perpendicular to mn, to the position p, which is vertically projected at p' on $m'n'$. This operation is identical with that represented in Figs. 56 and 57.

62. Problem 2. *To pass a plane through three given points not in the same right line.*

Analysis. Through either two of the points draw a right line. Through the third point draw another right line, either parallel to the first or intersecting it at any point. These two lines determine the plane, and their traces will be points in the corresponding traces of the plane.

Construction. In Figs. 64 and 65, C, D, and E are the given

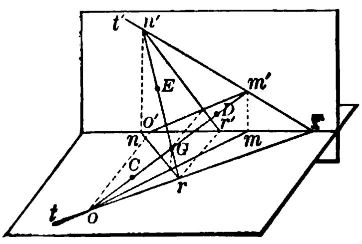


Fig. 64

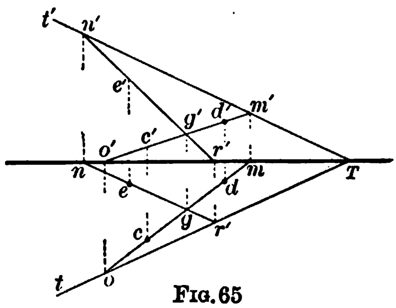


Fig. 65

points. Draw CD and produce it to pierce **H** at O, and **V** at M; then o is a point in the horizontal and m' a point in the vertical trace of the required plane. Join the third point E with any point

G of OM, and produce EG to pierce **H** and **V** at R and N; then r is another point in the horizontal and n' another point in the vertical trace. Therefore $n'm'$, or are the required traces, which, when produced, must meet in the ground line, unless they are parallel.

Note.—The direction of the second line EG should be so chosen that the distance between o and r, and also that between n' and m', shall be as great as possible.

63. The problems of drawing a plane through one right line and parallel to another, and of drawing a plane through a given point parallel to two given right lines, are scarcely more than variations of the preceding one; for, in the first case, we have already one line of the required plane and know the direction of another, which may be drawn through any given point of the given line; and in the second case, we know the directions of two lines of the required plane and have merely to draw them through a given point and find their traces. If either line be parallel to **AB**, the plane itself and both its traces will be parallel to the ground line. In this case a profile should be drawn, in addition to the projections on **H** and **V**.

64. PROBLEM 3. *To draw through a given point a plane perpendicular to a given right line.*

Analysis. The directions of the traces are known, being respectively perpendicular to the projections of the line (**43**). Draw through the given point a line parallel to either trace; this will be a line of the plane, and will pierce the other plane of projection in a point of the required trace upon that plane. This trace, being perpendicular to the corresponding projection of the line, may now be drawn; it will cut the ground line in a point of the remaining trace, of which the direction is also known.

Construction. Let P, Figs. 66 and 67, be the given point and MN the given line. Draw through P a line parallel to the horizontal trace of the required plane; its horizontal projection is po, perpendicular to mn, and its vertical projection is $p'o'$, parallel to **AB**. This is a line of the plane, and its vertical trace O is a point in the vertical trace of the plane. Therefore $t'o'T$, perpendicular

to $m'n'$, is that vertical trace, which cuts **AB** at T; and Tt, perpendicular to mn, is the horizontal trace.

N. B. If the projections of the given line coincide, as in Fig. 29, the traces of the plane will also coincide, like those of the plane tTt' in Fig. 40. If the given line lie in a profile plane, the required plane will be parallel to the ground line. Thus in Fig. 68, P is the given point, MN the given line; these are seen in their true relations to **H** and **V** in the profile, Fig. 69, where a per-

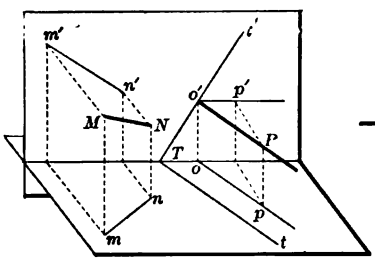


FIG. 66

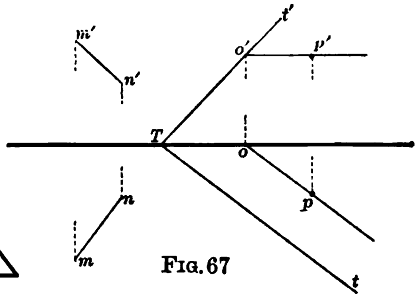


FIG. 67

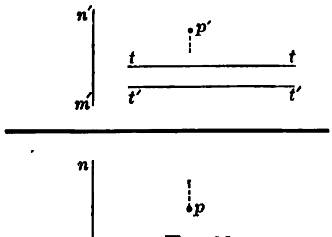


FIG. 68

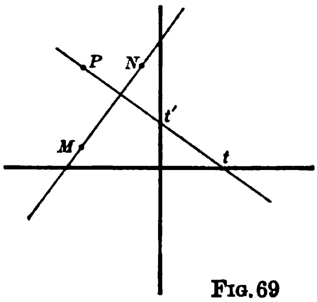


FIG. 69

pendicular to MN through P represents the required plane, cutting **V** in t' and **H** in t; these points are the profile projections of the traces tt, $t't$, in Fig. 68.

65. The reasoning in the analysis of this problem is precisely the same as that used (**41**) in reference to the drawing of a plane through a given point and parallel to a given plane, the construction of which was shown in Fig. 50. The gist of the argument is simply this, that when the directions of the traces are known, the location of a single point in either trace determines the plane, except when it is parallel to the ground line; in that case a point in each trace must be found.

66. Problem 4. *To find the intersection of two planes.*

Analysis. The intersection of the vertical traces will be one point, and the intersection of the horizontal traces will be another, in the required line, which is determined by those two points. If both planes are perpendicular to either of the principal planes, their traces on that plane will be parallel to each other and to the required line, which will pass through the intersection of the other two traces. If both planes are parallel to the ground line, the required line will be so likewise; it is determined by the intersection of the profile traces of the given planes.

Construction. In Figs. 70 and 71, *sSs'*, *tTt'* are the two planes.

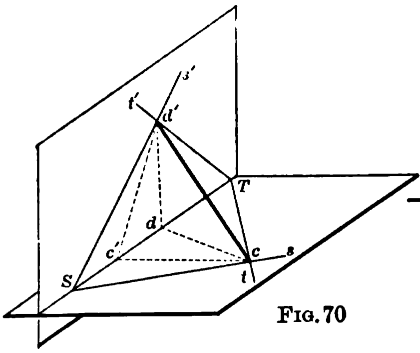


Fig. 70

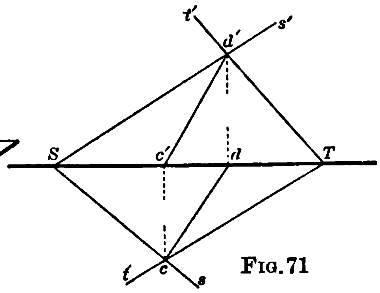


Fig. 71

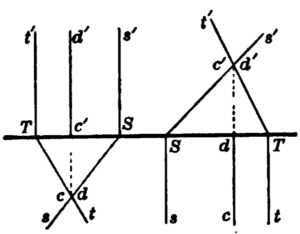


Fig. 72

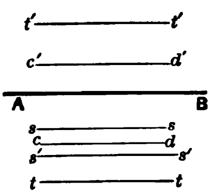


Fig. 74

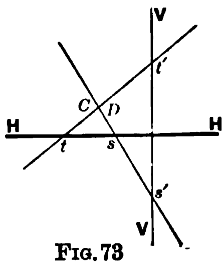


Fig. 73

The horizontal traces intersect at *c*, whose vertical projection is *c'* in **AB**, and the vertical traces intersect at *d'*, whose horizontal projection is *d* in **AB**; therefore *cd* is the horizontal and *c'd'* the vertical projection of the required intersection.

When both planes are vertical, their intersection is vertical and passes through the intersection of the horizontal traces, as shown in Fig. 72, at the left; when the planes are perpendicular to **V**, as

shown in the same figure at the right, their intersection is also perpendicular to **V** and passes through the intersection of the vertical traces.

When both planes, and consequently their intersection, are parallel to **AB**, the detached profile, Fig. 73, shows the condition of things with perfect distinctness; but the projection on the principal planes, Fig. 74, is by itself simply useless as a means of imparting information.

67. *Some Special Cases of the Above Problem.*—In Fig. 75 the horizontal traces do not intersect within the limits of the drawing; but one point, D, of the required line is determined by the intersection of the vertical traces. In order to ascertain its direction, draw an auxiliary plane lLl', parallel to tTt' and cutting sSs'

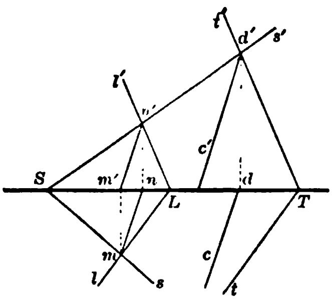


FIG. 75

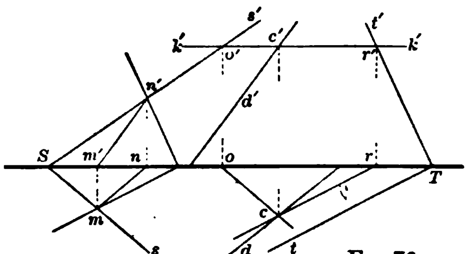


FIG. 76

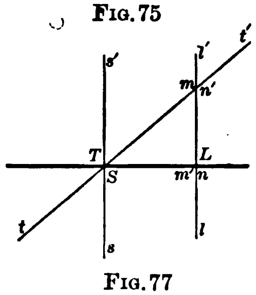


FIG. 77

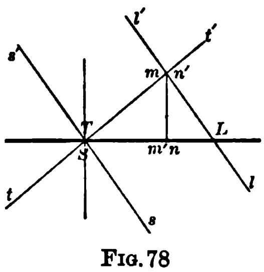


FIG. 78

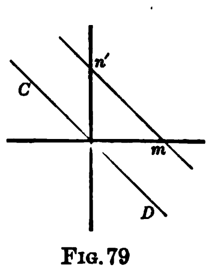


FIG. 79

in the line MN, found as in Fig. 71. This intersection is parallel to the one sought, of which, therefore, the vertical projection is $c'd'$, parallel to $m'n'$, and cd, parallel to mn, is its horizontal projection.

In Fig. 76 the intersections of the vertical traces and of the horizontal traces are both inaccessible. Draw an auxiliary horizontal plane, of which $k'k'$ is the trace. This cuts the plane tTt' in a

line of which one point is r' on **V**, horizontally projected at r on **AB**; this line, being horizontal, is parallel to the horizontal trace and its horizontal projection is therefore drawn through r and parallel to Tt. The auxiliary plane also cuts sSs' in a line whose horizontal projection is drawn through o, parallel to Ss. These two lines, one in each given plane, cut each other in a point of which the horizontal projection is c, and the vertical projection is c' on $k'k'$. Thus one point in the required line is determined, and its direction is ascertained as in Fig. 75.

In Fig. 77 the traces of the plane tTt coincide as in Fig. 40, and sSs' is a profile plane cutting the ground line at the same point; in Fig. 78 both planes are oblique, with coincident traces. Drawing in each case an auxiliary plane parallel to sSs, as in Fig. 75, the line MN cut from the plane tTt' is parallel to the required intersection. And in each case this line pierces **H** behind **V**, and **V** above **H**, in points equally distant from **AB**; it therefore crosses the second angle, as shown in the profile, Fig. 79, and is equally inclined to **H** and **V**. The required line CD, being parallel to MN and intersecting **AB**, therefore lies in a profile plane and bisects the first and third angles; its projections in Fig. 77 coincide in ss', and in Fig. 78 they coincide in a line through T, perpendicular to **AB**.

68. Problem 5. *To find the point in which a given right line pierces a given plane.*

Analysis. Pass any plane through the given line and find its intersection with the given plane. This line will cut the given line in the required point.

Construction. In Figs. 80 and 81, MN is the given line, tTt' the given plane. Since any plane containing MN will serve our purpose, we use for convenience one of its projecting planes; in this case the horizontal. Its horizontal trace coincides with the horizontal projection of the line, and its vertical trace is perpendicular to **AB**; it intersects tTt' in the line CD, whose vertical projection $c'd'$ cuts the vertical projection $m'n'$ in o', the vertical projection of the required point; the horizontal projection is o, on cd.

N. B. Had the intersection at o' been very acute, the determination would have been less reliable; and a better result might have been obtained by using the vertical projecting plane, thus de-

termining first the horizontal projection o of the required point. It is not certain that this would happen, since if the line were but slightly inclined to the plane, both these intersections would be acute; in which case both determinations should be made, and if

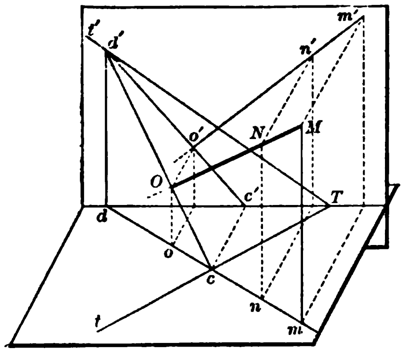


FIG. 80

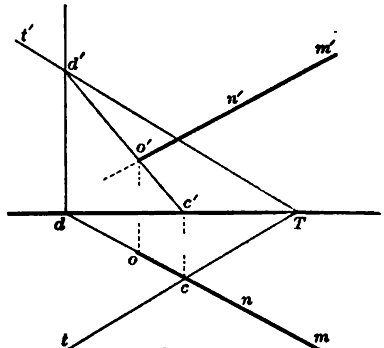


FIG. 81

they do not agree, a mean between them may be taken as the correct result.

69. The preceding construction involves the use of the traces of the given plane: but if two lines of a plane are given, it is not necessary to find the traces in order to determine the point in which it is pierced by a third line. Thus in Fig. 82, let it be required to

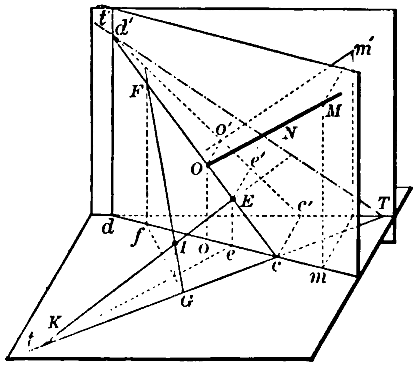


FIG. 82

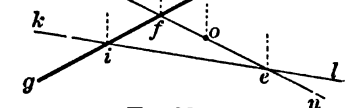


FIG. 83

find the point in which the line MN pierces the plane determined by the two intersecting lines KI, GI. Using again for convenience the horizontal projecting plane mdd' of the given line, it cuts

KI in *E*, and *GI* in *F*; *EF* therefore lies in both planes, and the point *O* in which it intersects *MN*, is the required point. The traces of the given plane are shown in this pictorial representation, for the purpose of calling attention to the fact that *EF* is merely a portion of the same line of intersection *CD*, which was determined in Fig. 80 by means of the traces of the two planes.

The construction in projection is given in Fig. 83, where *GR*, *KL*, intersecting at *I*, determine a plane, and it is required to find the point in which this plane is pierced by the line *MN*. The horizontal trace *mn*, of the horizontal projecting plane, must contain the horizontal projections of all lines and points that lie in it, because the plane is vertical. And *mn* cuts *gr* at *f*, which is the horizontal projection of a point on *GR*, whose vertical projection is *f'* on *g'r'*. Similarly, the line *KL* is seen to pierce the projecting plane in a point whose horizontal projection is *e*, and whose vertical projection is *e'* on *k'l'*. Consequently *e'f'* is the vertical projection of the portion of the line of intersection thus determined; it cuts *m'n'* in *o'*, the vertical projection of the required point, and the horizontal projection is *o* on *mn*.

70. *Some Special Cases of the Above Problem.* In Fig. 84, the given plane is parallel to **AB**, and the projections of the

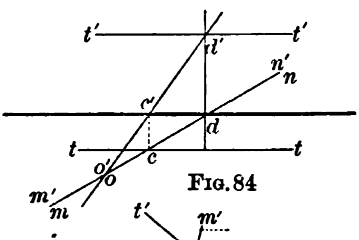


FIG. 84

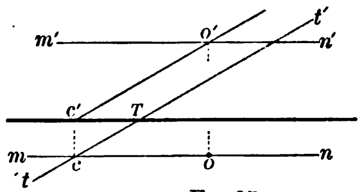


FIG. 85

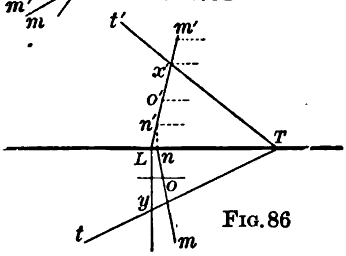


FIG. 86

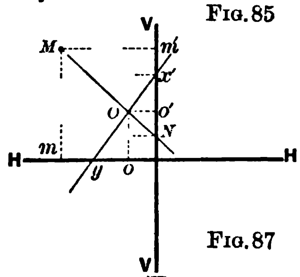


FIG. 87

given line *MN* coincide. The horizontal trace of the horizontal projecting plane cuts *tt* at *c*, vertically projected at *c'* in **AB**; its

vertical trace cuts $t't'$ at d', of which d is the horizontal projection; and $c'd'$ intersects $m'n'$ in o', the vertical projection, which coincides with o, the horizontal projection of the required point.

In Fig. 85, the traces of tTt' coincide, and MN is parallel to **AB**. The horizontal projecting plane cuts tTt' in a line parallel to **V** and therefore to Tt'; one point of this line is determined by the intersection of the horizontal traces at c, vertically projected at c' in **AB**, and its vertical projection $c'o'$ cuts $m'n'$ at o', of which o on mn is the horizontal projection.

In Fig. 86, the two projections of MN are nearly perpendicular to **AB**. In such cases the direct determinations by the methods before explained are apt to be very unreliable on account of the acuteness of the intersections: and the profile may be used to great advantage in the manner here illustrated. In this instance the vertical projecting plane of the given line has been used; its vertical trace cuts Tt' at x', and the ground line at L; its horizontal trace is perpendicular to **AB** and cuts Tt at y. The line of intersection will therefore pass through x' on **V** and y on **H**; but its projection on the latter is not drawn. In drawing the profile, Fig. 87, x' is projected horizontally across from Fig. 86, and the distance of y from VV is equal to Ly in the horizontal projection; then $x'y$ represents the line of intersection. In this particular case MN pierces **V** at N, therefore n' is projected directly across to VV; the altitude of M is the same in both views, and so is its distance from **V**; and MN in the profile intersects $x'y$ in O, which being projected back to $m'n'$ in Fig. 86, determines o' the vertical projection of the required point. The distance of O from **V** is seen in the profile; and drawing in Fig. 86 a parallel to **AB** at that distance from it, the horizontal projection o is determined much more accurately than it could be by drawing through o' a perpendicular to **AB**.

Should the given line lie in a plane perpendicular to **AB**, the construction of a profile is of course a necessity.

71. PROBLEM 6. *To find the distance of a given point from a given plane.*

Analysis. 1. Draw through the point a perpendicular to the

plane. 2. Find the point in which it pierces the plane. 3. Find the distance between this point and the given point.

Construction. In Fig. 88, let P be the given point, tTt' the given plane. Draw through P a perpendicular to tTt' as in Fig.

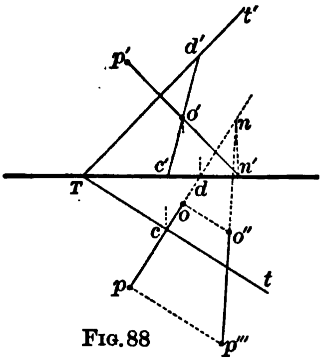


FIG. 88

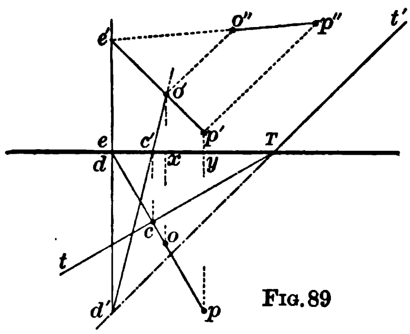


FIG. 89

52. Find the point O in which it pierces the plane as in Fig. 81. Find the true length of PO as in Fig. 63.

In Fig. 89, the direction of the vertical trace Tt' is the same as in Fig. 88, but that of the horizontal trace Tt is different. In consequence of this change, the vertical trace of the horizontal projecting plane cuts that of the given plane at a point d' below **AB** instead of above it as before, and $d'c'$ must be produced to determine o'. PO is here revolved into **V** instead of **H**; it pierces **V** at a point of which the horizontal projection is e, and the vertical projection is e' on $p'o'$ produced; and since e' is on the axis, it remains fixed, and the prolongation of $p''o''$ passes through it.

When the given plane is parallel to **AB**, the required distance is found directly by constructing a profile.

72. PROBLEM 7. *To project a given right line upon a given plane.*

Analysis. Through any point of the given line, draw a perpendicular to the given plane: these two lines determine a second plane, perpendicular to the first. The intersection of these two planes is the required projection.

Construction. In Fig. 90, let KG be the given line, tt the horizontal trace, and $t't'$ the vertical trace of the given plane. From any point P on KG, draw a perpendicular to the plane; the traces of

this perpendicular are X and Z, and the traces of the given line are U and Y. Therefore uz is the horizontal and $x'y'$ is the vertical trace of the plane sSs', determined by the given line KG and the projecting perpendicular PX. This plane cuts the given plane in the line CD, which is the required projection.

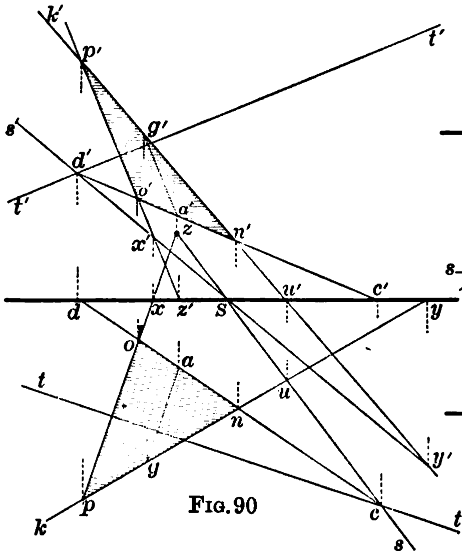


FIG. 90

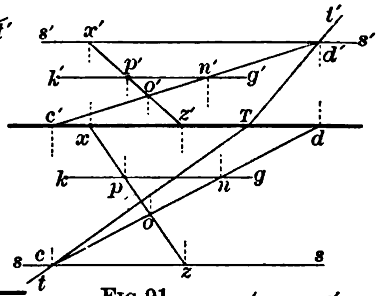


FIG. 91

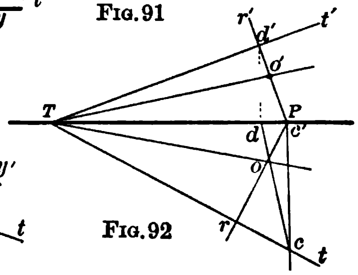


FIG. 92

73. This intersection CD evidently must contain the point N, in which the given line pierces the plane tTt', and also the point O, which is the foot of the perpendicular let fall upon the plane from the point P; and ON is the projection of the hypothenuse PN of the right-angled triangle PON.

The points O and N might have been found as in Problem 5, without determining the traces of sSs'. And if the projection of a definite portion of the line, as for instance PG in the figure, is required, two perpendiculars, as PO, GA, may be drawn, and the points of penetration, O and A, found in the same way; indeed, this may be necessary, if the line is but slightly inclined to the plane. The three methods are identical in principle, and the selection must depend upon considerations of convenience, determined by the given conditions in any particular case.

If the given line be parallel to the given plane, its projection on

that plane will be parallel to the line itself (**14**). Therefore the required projections will be parallel to those of the given line, and the determination of one point in each is sufficient.

74. *Some Special Cases of the Above Problem.* In Fig. 91, tTt' is the given plane; and the given line KG is parallel to **AB**. Draw through any point P on GK a perpendicular to tTt'; it pierces the horizontal plane in Z and the vertical plane in X. The plane of these two lines is parallel to **AB**, therefore its traces are sz and $s'x'$, also parallel to **AB**; and it cuts tTt' in the line CD, the required projection.

In Fig. 92, tTt' is the given plane; it is required to project the ground line on it. From any point P on **AB** draw Pr perpendicular to Tt and Pr' perpendicular to Tt'; these are the projections of a line perpendicular to the plane. The vertical projecting plane of this line cuts tTt' in the line CD, which intersects PR in O, the projection of the point P upon the given plane. That plane cuts **AB** in the point T; consequently To is the horizontal and To' is the vertical projection of the required line.

In Fig. 93, the given line is inclined to both planes, piercing **H** in the point U and **V** in the point Y. The given plane being parallel to the ground line, the perpendicular to it from the point P

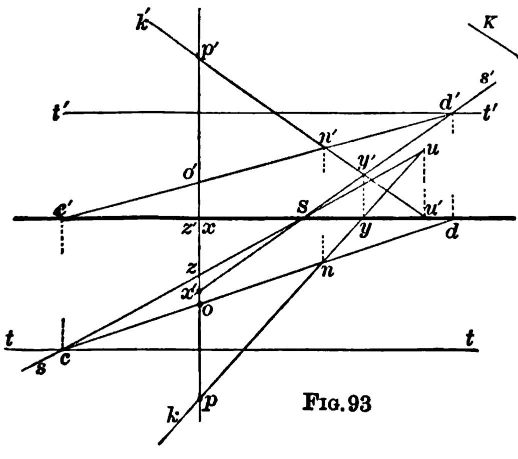


FIG. 93

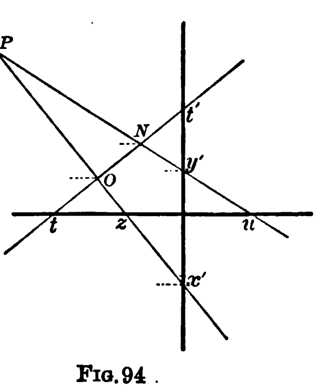


FIG. 94.

will lie in a plane perpendicular to **AB**; its traces are readily determined by drawing the profile, Fig. 94; then setting off z and x' in Fig. 93 at the distances from **AB** thus found, we have, as before,

uz for the horizontal and *x'y'* for the vertical trace of the plane *sSs'*, which cuts *tTt'* in the line *CD*, the required projection.

75. PROBLEM 8. *To find the distance of a given point from a given line.*

First Method. **Analysis.** 1. Through the given point pass a plane perpendicular to the given line. 2. Find the point in which the given line pierces this plane. 3. Find the distance between this point and the given point.

Construction. In Fig 95, let *P* be the given point, *KG* the given line. Draw through *P* a plane *tTt'* perpendicular to *KG*, as

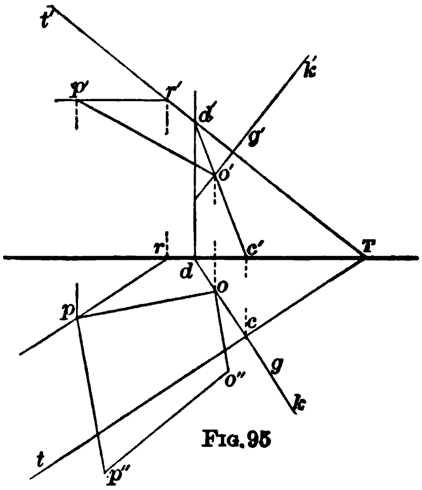


FIG. 95

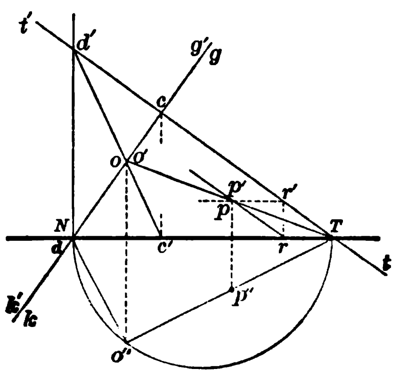


FIG. 96

in Fig. 67. Find the point *O*, in which *KG* pierces *tTt'*, as in Fig. 81. Find the length of *PO* as in Fig. 63.

Special Case. In Fig. 96, the two projections of *P* coincide, as do those of *KG*; consequently the traces of the perpendicular plane *tTt'* also coincide. The horizontal projecting plane of *KG* cuts *tTt'* in the line *CD*, which intersects *KG* in *O*. The line *KG* lies in a plane bisecting the second and fourth angles, and cuts **AB** at *N*; the line *OP* also lies in that bisecting plane, and since it is at the same time a line of the plane *tTt'*, it will when produced cut **AB** at *T*. Therefore *NT* is the hypothenuse of the right-angled triangle *TON*; and when this triangle is revolved about **AB** into either **V** or **H**, *O* will fall at *o''* on the circumference of a semicircle of which *NT* is the diameter, and *P* falls at *p''* on *o''T*.

76. *Second Method.* **Analysis.** Through the given point draw a line either parallel to or intersecting the given line. Revolve the plane of these two lines about one of its traces into the corresponding plane of projection; the line and point will then be seen in their true relative positions. A perpendicular from this revolved position of the point to that of the line will be the required distance.

Construction. In Fig. 97, P is the given point, KG the given line. Draw through P a parallel to KG; it pierces V in N, and KG pierces it in M. Revolve this plane about its vertical trace

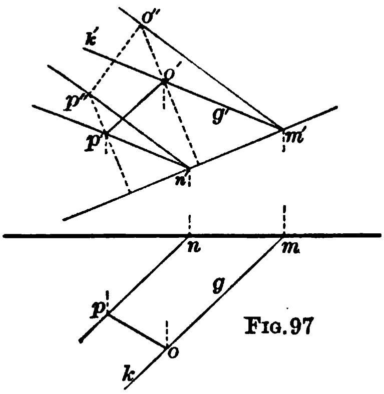


FIG. 97

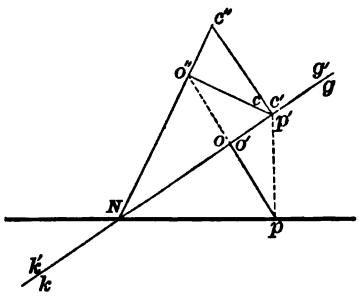


FIG. 98

$n'm'$ into V; P goes to p'', and as n' remains fixed, being in the axis, $p''n'$ is the revolved position of the second line. And since the two lines are parallel, a line through m', parallel to $p''n'$, is the revolved position of the given line, and $p''o''$, perpendicular to it, is the actual distance required.

In the counter-revolution, p'' returns to p', and o'' goes, in a direction perpendicular to the axis $m'n'$, to the position o' on $m'k'$. This is the vertical, and o on km is the horizontal, projection of O, the foot of the perpendicular drawn from the given point P to the given line.

Special Case. In Fig. 98, the projections of the given line KG coincide, and the given point P is in the vertical plane. There is a point on KG whose projections, c and c', coincide with p'. The vertical projecting line of this point therefore passes through the given point, and, with the given line, determines a plane of which $c'N$ is the vertical trace. Revolving this plane about this trace

into V, C goes to c'', $c''N$ is the revolved position of the given line, and p' is stationary. Therefore $p'o''$ perpendicular to $c''N$ is the required distance. In the counter-revolution c'' returns to c', o'' goes to o' on $k'g'$, and po, $p'o'$, are the projections of the perpendicular.

77. PROBLEM 9. *To find the angle between two lines, and to divide it.*

Analysis. If the plane of the two lines be revolved about one of its traces, or a line parallel thereto, until it coincides with or is parallel to the corresponding plane of projection, the angle will be seen in its true size and may be subdivided.

Otherwise: If a supplementary projection be made upon a plane parallel to that determined by the given lines, the angle will appear in its true size.

Construction. In Fig. 99, the two lines which intersect in O pierce H in the points P and N. Revolving them about the horizontal trace np of their plane into H, their intersection O goes to o'', the distance xo'' being equal to the hypothenuse of a triangle of which ox is the base and $o'y$ the altitude. Thus the angle is seen in its true size as $no''p$; if it be now required to divide it, for instance into two equal parts, the bisecting line cuts np at r, which in the counter-revolution will remain stationary: therefore or, $o'r'$, are the projections of the bisector.

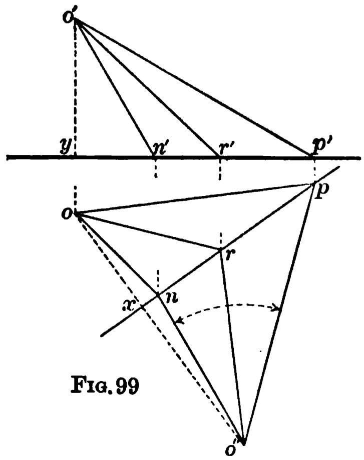


FIG. 99

78. In Fig. 100, the horizontal trace of OP is inaccessible. Draw therefore an auxiliary horizontal plane hh, cutting the given lines in C and D, and revolve the lines about cd until they are parallel to H; in this revolution O goes to o'', the distance xo'' being equal to the hypothenuse of a triangle having for its base ox, and for its altitude $o'y$, the distance of the point from the plane hh. Then $co''d$ is the angle in its true size, the bisector cuts cd in s, and its projections after the counter-revolution are os, $o's'$.

In Fig. 101, the line OP is but slightly inclined to **AB**, so that it is inconvenient to find where it pierces either of the principal

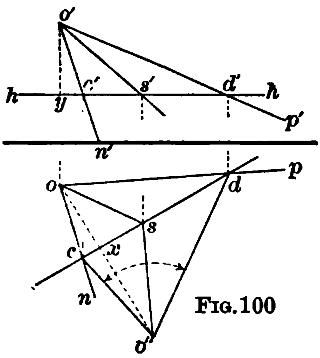


Fig. 100

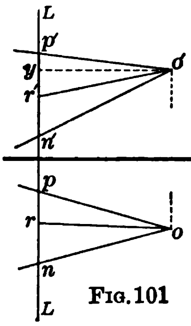


Fig. 101

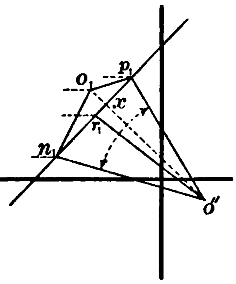


Fig. 102

planes. In this event, draw a plane LL, perpendicular to **AB**, cutting the given lines in the points P and N. Constructing the profile, Fig. 102, the vertex of the angle appears at o_1, and $n_1 p_1$ is the trace of the plane of the two lines upon the plane LL. Revolve the lines about this trace into that plane; o_1 goes to o'', the distance xo'' being equal to the hypothenuse of a triangle whose base is $o_1 x$, the distance of the projection of the point from the axis, and the altitude is $o'y$, the distance of the point from the plane in which the axis lies; $n_1 o'' p_1$ is the angle in its true size.

79. In Fig. 103, the line OP is perpendicular to **V**; and so, consequently, is the plane of the two given lines. A supplementary projection upon a plane parallel to this plane at once shows the angle in its true size. In this projection, the vertical plane will appear (**55**) as a line VV parallel to $o'n'$, and OP as a line $o_1 p_1$ perpendicular to VV. Draw any line NM cutting both the given lines: the points upon OP will all be vertically projected in o', and the vertical projection of N is n'. These points are all projected perpendicularly toward VV, and the distances of m_1, n_1, o_1, from VV, in the new projection, are equal to the distances of m, n, and o from **AB** in the horizontal projection. The angle, being now seen in its true size, may be bisected as before, the bisector cutting $m_1 n_1$ at r_1: this point is projected back to r', and thence to r, giving or as the horizontal projection of the bisecting line.

80. Such projections as this are not only of great convenience in many of the operations in abstract descriptive geometry, but they are of every-day occurrence in making mechanical drawings for industrial purposes. Indeed, the construction in Fig. 103 is identi-

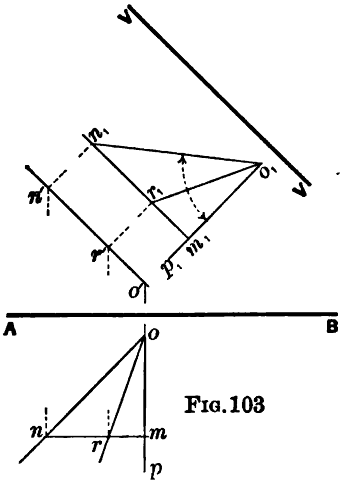


FIG. 103

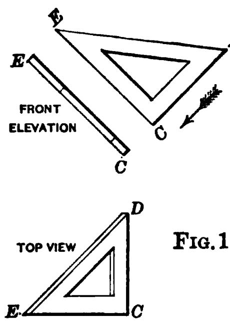


FIG. 104

cal with that used in making the three views of the draughtsman's triangle, shown in Fig. 104. This implement is seen edgewise in the front view, where the side CD is perpendicular to the paper; and it is foreshortened in the top view. Now in order to exhibit the true form and size, no practical man will "revolve it about its side until it becomes horizontal," or otherwise disturb it: on the contrary, leaving it at rest, a view is drawn, looking perpendicularly against it as indicated by the arrow. There is no need of using a reference plane, like VV in Fig. 103, in ordinary cases, because its place is supplied by lines or planes of the object itself: thus, the distances from EC in this instance are the same in the original top view and in the third or supplementary view.

81. *Special Case of the Above Problem.* To find the angle included between the vertical and horizontal traces of a given plane: Revolve the vertical trace about the horizontal trace into **H**. In Fig. 105, let O be any point on the vertical trace of the plane tTt'; its vertical projection is o' on Tt', and its horizontal projection is o on **AB**. In the revolution, O goes to o'', on a perpendicular to Tt through o, the distance xo'' being equal to the hypoth-

enuse of a triangle of which ox, oo', are the base and the altitude. Also, To'' is equal to To', since the distance To' is seen in its true length, and remains unchanged during the revolution: and tTo'' is the required angle.

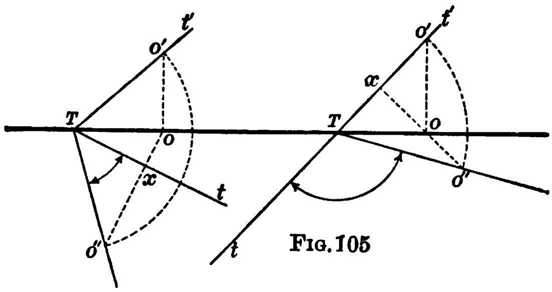


FIG. 105

82. PROBLEM 10. *To find the angle between a given line and a given plane.*

Analysis. The angle which a line makes with a plane is the same as that included between the line itself and its projection on the plane.

From any point of the line, draw a perpendicular to the plane. From any other point of the given line, draw a perpendicular to the second line. This third line will be parallel to the projection of the given line upon the plane, and the angle which it makes with the given line is the one required.

Construction. The pictoral representation, Fig. 106, illustrates the analysis; the given line MN pierces the given plane tTt' at N,

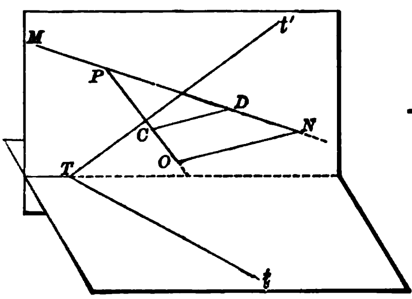


FIG. 106

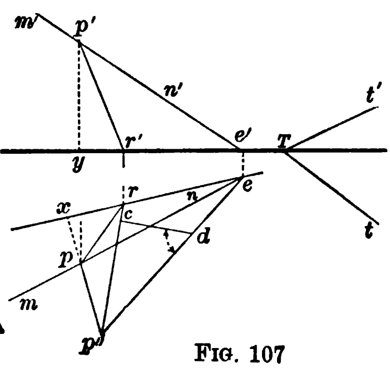


FIG. 107

the perpendicular from P pierces it at O, NO is the projection of NP, and PNO is the required angle. But DC, perpendicular to

PO, is parallel to NO, and the angle PDC is equal to the angle PNO. Consequently it is not *necessary* to find either the point N or the point O, but DC may be drawn anywhere in the projecting plane determined by MN and PO.

Now in Fig. 107, from any point P on the given line MN, draw a perpendicular to the given plane tTt'; it pierces **H** at R, and MN pierces **H** at E. Revolve the plane of these two lines into **H** about its horizontal trace er; P goes to p'', and $rp''e$ is the angle at P in its true size. From any point d on $p''e$, draw dc perpendicular to $p''r$, the revolved position of the projecting line PR; then $p''dc$ is the required angle.

Note. Should the given line be inconveniently situated, any parallel to it may be used instead.

83. The determination of the angle may, however, sometimes be facilitated by finding the actual projection of the line upon the plane. Thus in Fig. 108, the given plane is parallel to **AB**; and

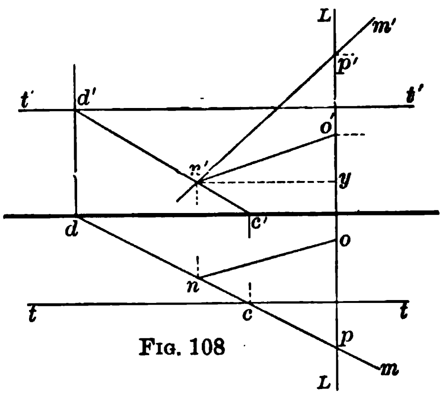


FIG. 108

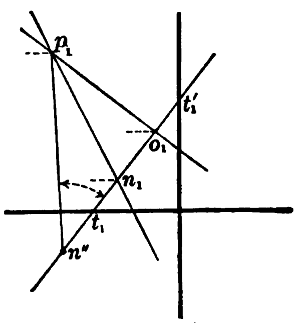


FIG. 109

the construction is as follows: Draw a plane LL perpendicular to **AB**; the given line MN pierces this plane at P, and the given plane at N. Construct the profile, Fig. 109: the given plane is here seen edgewise as the line t_1t_1', to which n' is projected at n_1, and the point P is found at p_1. From p_1 let fall upon t_1t_1' the perpendicular p_1o_1, and project o_1 back to o'; the distance of o below **AB** in Fig. 108 is equal to the distance of o_1 in the profile in front of **V**. We have thus the projection NO of the line NP upon the given plane; and it is seen that the line PO lies in the plane LL.

Revolve N about p_1o_1 into this profile plane; N falls at n'', the distance o_1n'' being equal to the hypothenuse of a triangle whose base is o_1n_1, and altitude $n'y$; and $p_1n''o_1$ is the angle sought.

84. *Special Cases of the Above Problem.* In Fig. 110, it is required to find the angle made by the plane tTt' with the ground

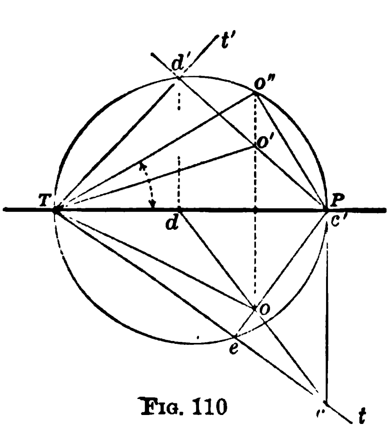


FIG. 110

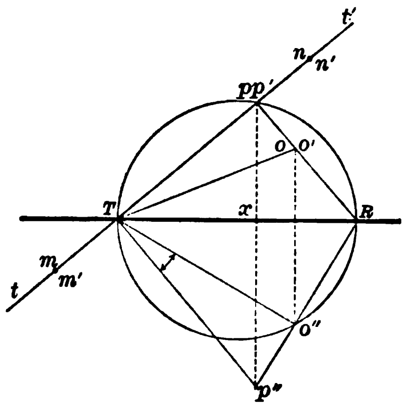


FIG. 111

line. From any point P on **AB**, draw Pe, Pd', respectively perpendicular to Tt and Tt'; these are the projections of a perpendicular to the plane, and O is the point in which it pierces the plane. Therefore TO is the projection of **AB** upon tTt', and PO is perpendicular to it. Revolve the plane of these two lines about **AB** into **V**; O falls at o'', and PTo'' is the required angle.

The points e, d', and o'' must lie upon the circumference of a circle whose diameter is TP; and the construction may be facilitated by drawing the circle first.

In Fig. 111, the traces of a plane tTt', and the projections of a line MN, coincide in one line inclined to **AB**; it is required to find the angle between the line and the plane. From any point P on MN, draw a perpendicular to the plane; its projections coincide, therefore this perpendicular cuts **AB** at R. Revolve the plane of PT and PR about **AB** into **H**; P falls at p'', the distance xp'' being equal to the hypothenuse of a triangle whose base and altitude are each equal to px; draw To'' perpendicular to $p''R$, then $p''To''$ is the required angle.

Evidently o'' is also the revolved position of O, the foot of a

perpendicular let fall from R to the plane tTt', and To'' the revolved position of the projection of **AB** upon that plane. Therefore RTo'' is the angle made by the ground line with the given plane, and RTp'' is the angle between **AB** and the given line.

85. PROBLEM 11. *To find the angle between two given planes.*

Analysis. Any plane perpendicular to the intersection of the given planes will be perpendicular to both, and will cut each of them in a line. These two lines will be perpendicular to the intersection at the same point, and the angle between them is the required angle.

Construction. This might be accomplished by applying the preceding problems, thus: 1. Find the intersection of the given planes as in Problem 4. 2. Through any point of that line draw a plane perpendicular to it, as in Problem 3. 3. Find the intersection of this plane with each of the given planes, as in Problem 4. 4. Find the angle between these two lines, as in Problem 9.

But a neater and less laborious process is pictorially represented in Fig. 112, where CKD, CID, are the given planes, and CED is the horizontal projecting plane of CD their line of intersection. The plane MPN is perpendicular to CD, therefore its horizontal

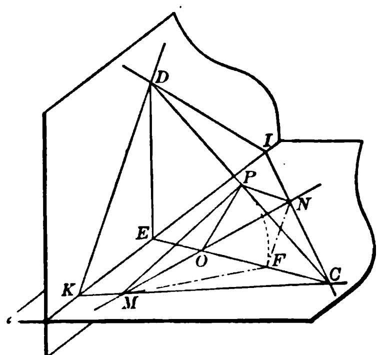


FIG. 112

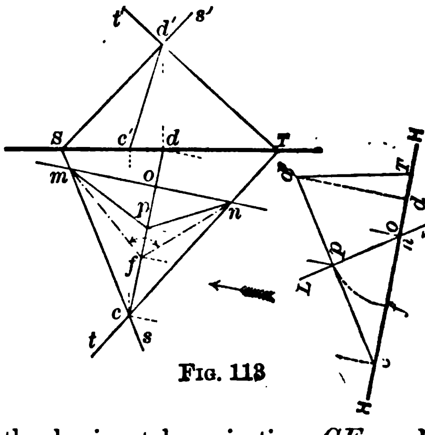


FIG. 113

trace MN is perpendicular to the horizontal projection CE, and cuts it at O. MN is also perpendicular to the vertical line through O, which lies in the projecting plane. Therefore MN is perpendicular to the plane CED, and consequently to OP, which is thus shown to be the true distance from O to the vertex of the angle.

Revolving MPN about MN into **H**, P falls at F on CE, OF being equal to OP, and MFN is the required angle.

86. This construction in projection is given in Fig. 113, where tTt', sSs', are the given planes, and CD is their intersection. Construct a supplementary view, looking perpendicularly against the horizontal projecting plane of CD, as shown by the arrow; in this view the horizontal plane is seen as $H'H'$, in which c lies: the altitude dd' is the same as in the vertical projection, and the line CD is thus seen in its true length and inclination. Draw LL perpendicular to cd' in this view; it is the plane MPN in Fig. 112, seen edgewise, and cuts the horizontal plane in the line appearing as mn, perpendicular to cd, in the horizontal projection. It also cuts cd' in a point p, which may be projected back to cd, thus determining pm, pn, the horizontal projections of the lines including the angle sought. But for the purpose of measuring the angle, this is not necessary, since op in the supplementary view is the true distance from O to the vertex; and setting this distance off as of on cd in the horizontal projection, we have mfn as the angle in its true size.

In Fig. 114, the traces of one plane, tTt', are coincident, and the other plane, sSs', is parallel to **AB**. The construction is the

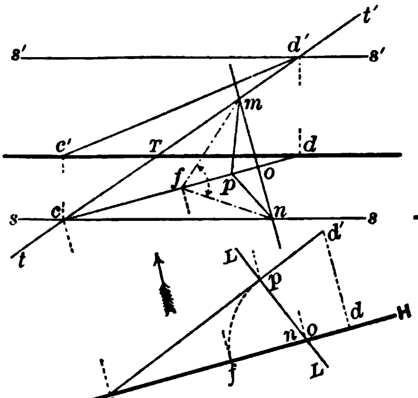


Fig. 114

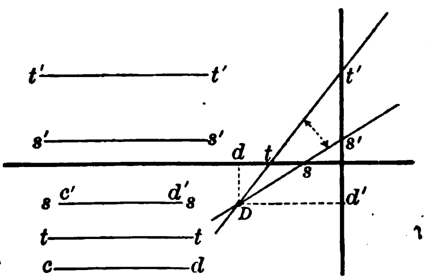


Fig. 115

same as in Fig. 113, and the two diagrams being lettered to correspond throughout, no further explanation is required.

In Fig. 115, both planes are parallel to **AB**; the profile tells the whole story, and the projections on **H** and **V** are simply useless.

87. To find the angle made by a given plane with either plane of projection.

In Fig. 116, tTt' is the given plane; to measure its inclination to **H**, draw a plane sSs' perpendicular to the horizontal trace, as in the diagram at the left, and revolve its intersection CD with

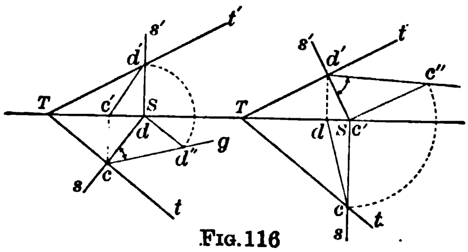


FIG. 116

the given plane, about cd into **H**; D falls at d'', the distance dd'' being equal to dd', and dcd'' is the required angle. To find the angle made with the vertical plane, draw sSs' perpendicular to the vertical trace, as in the diagram at the right, and revolve the intersection into **V** about its vertical projection: C falls at c'', $c'c''$ being equal to $c'c$, and $c'd'c''$ is the required angle.

Conversely: Given one trace and the angle with the corresponding plane of projection, to find the other trace. This is done by simply reversing the preceding operation. Thus, let the horizontal trace and the angle with **H** be given; then in the diagram at the left, draw sS perpendicular to Tt and Ss' perpendicular to **AB**: make the angle Scg equal to the given angle, draw at S a perpendicular to sS, cutting cg in d''. Then set up Sd' equal to Sd'', and Tt' drawn through d' will be the required vertical trace.

88. *Special Case of the Above Problem.* In Fig. 117, it is required to find the angle made by the oblique plane tTt' with the profile plane sSs'. Drawing the profile, Fig. 118, the line of intersection CD is seen in its true length and inclination as cd', and the plane LL perpendicular to it is parallel to **AB**. The traces of this plane in Fig. 117 are ll, $l'l'$; it cuts the plane tTt' in the line PN. The lines PO, PM, of Fig. 112, in this case coincide in one line, cut from the profile plane sSs' by the plane LL, and seen in its true length as $o'p_1$ in Fig. 118. Revolve LL about its vertical trace into **V**; P falls at f, and Sfn' is the required angle.

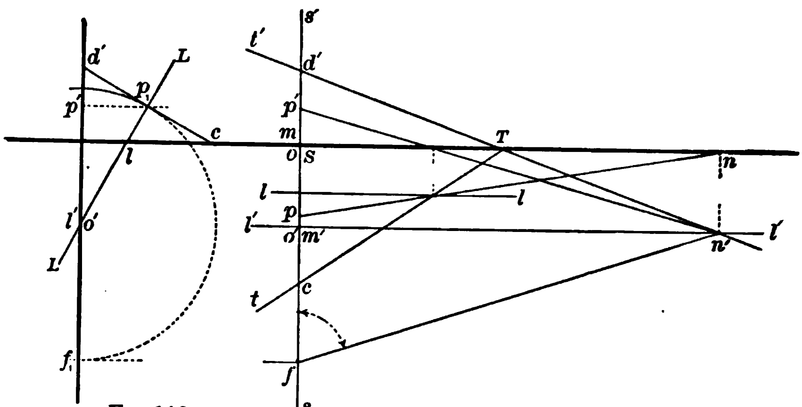


FIG. 118 FIG. 117

89. PROBLEM 12. *To find the common perpendicular of two lines not in the same plane.*

Analysis. If the two lines be projected upon a plane parallel to both, the projections will be respectively parallel to the lines themselves, and will intersect in a point. A perpendicular to the plane at this point, being perpendicular to both projections and therefore to each line, will cut them both; the portion intercepted between them is the required common perpendicular.

This is illustrated in Fig. 119, where CD, MN, are the two lines; their projections cd, mn, upon the parallel plane, intersect at E; the intersection of their projecting planes is the perpendicular at E, and the intercept PO is the required least distance between the given lines. If the plane approach the given lines, the projecting perpendiculars Mm, Nn, will be reduced in length, until, as in Fig. 120, they disappear, and MN lies in the plane, which is parallel to CD, and its intersection with cd at once determines the point P. Upon this is based one method of construction, which consists of the following steps, viz.:

Construction. 1. Through any point of one line, draw a parallel to the other, and find the traces of the plane thus determined.

2. Through any point of the second line, draw a perpendicular to this plane, and find the point in which it pierces the plane.

3. Through this point draw a parallel to the second line; it is the projection of that line upon the plane.

4. At the intersection of this projection with the first line, erect a perpendicular to the plane. It will cut the second line, and is the common perpendicular.

5. Determine the length of the intercept.

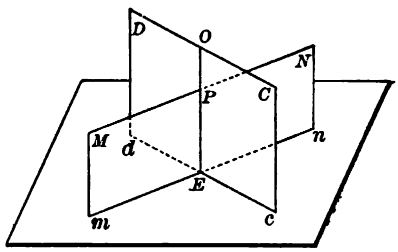


Fig. 119

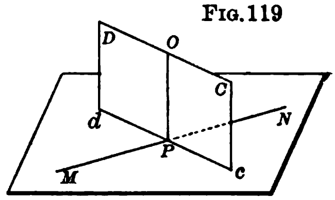


Fig. 120

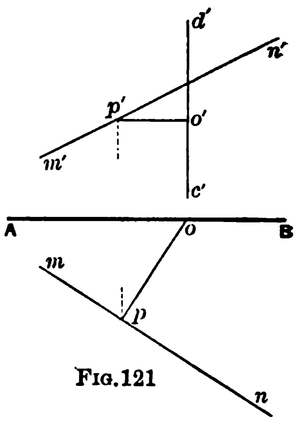


Fig. 121

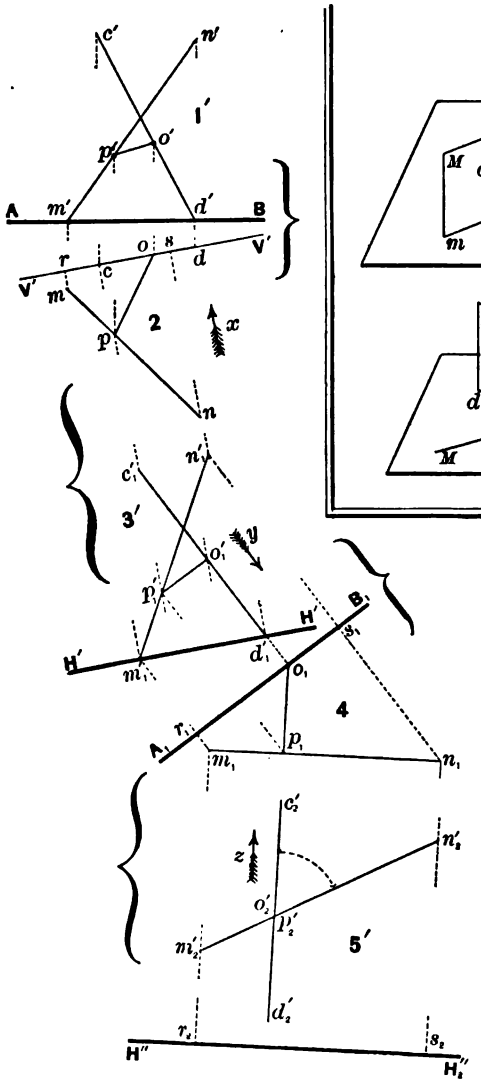


Fig. 122

90. In the execution of the above processes, the representations are all made on the principal planes of projection. But

the result may be attained in a far more direct and practical manner by the use of other planes. As a preliminary to the explanation, let $c'd'$, Fig. 121, be the vertical projection of a vertical line in **V**; its horizontal projection is o in **AB**: and let MN be an oblique line in another plane. Let op, perpendicular to mn, be the horizontal projection of a horizontal line intersecting CD and MN; its vertical projection is $p'o'$ parallel to **AB**. The line PO, being horizontal, is perpendicular to CD; and being perpendicular to the horizontal projecting plane of MN, it is also perpendicular to that line; moreover, it is seen in its true length as op in the horizontal projection.

91. Now in Fig. 122, the group (**1**′, **2**) contains the vertical and the horizontal projections of two lines CD, MN, not in the same plane; it is required to find their common perpendicular. Make a supplementary projection, **3**′, looking perpendicularly, as shown by the arrow x, against the horizontal projecting plane $V'V'$ of the line CD. In this projection the horizontal plane appears as $H'H'$ parallel to $V'V'$, and the distances of c_1', n_1', above $H'H'$ are the same as those of c', n', above **AB** in the original vertical projection.

Now, retaining the same vertical plane, make a projection, **4**, looking in a direction parallel to CD, as shown by the arrow y. For this purpose draw a new ground line, $\mathbf{A_1B_1}$, perpendicular to $c_1'd_1'$; in this projection CD will appear as a point o_1 in the new ground line, and the distances m_1r_1, n_1s_1, from $\mathbf{A_1B_1}$, are equal to the distances mr, ns, from $V'V'$ in the original horizontal projection. Holding the page so that $\mathbf{A_1B_1}$ is horizontal, it will be perceived that the group (**3**′, **4**) is identical with Fig. 121; and o_1p_1, perpendicular to m_1n_1, is the true length of the required line, which is projected in **3**′ as $o_1'p_1'$. The points o_1', p_1', are projected back from **3**′ to o, p, in **2**, and thence to o', p', in **1**′, thus determining the projections of OP, the common perpendicular, in the original position.

92. Going one step farther, make another projection, **5**′, looking, as shown by the arrow z, perpendicularly against the horizontal projecting plane m_1n_1 in **4**. In this last view the horizontal plane appears as $H''H''$ parallel to m_1n_1; and the distances above

it, as s_2n_2', r_2m_2', etc., are the same as the distances s_1n_1', r_1m_1', etc., above $\mathbf{A_1B_1}$ in the vertical projection **3'**.

In this view the line CD appears as $c_3'd_3'$ perpendicular to $H''H''$, and both it and the line MN are seen in their true lengths. What is practically of equal importance, the angle between the projections of the two lines on a plane parallel to both is also seen in its true size. Whereas, after executing the construction given in (**89**), we should be no nearer to the determination of this angle than we were before; and without knowing it, the relative positions of the two given lines is not fully defined: in fact, for practical purposes it is absolutely essential that it should be known.

93. By turning the page so that the lines **AB**, $H'H'$, $\mathbf{A_1B_1}$, $H''H''$, in succession, are brought into a horizontal position, it will be perceived that each of the groups (**1'**, **2**), (**3'**, **2**), (**3'**, **4**), and (**5'**, **4**) consists of a vertical and a horizontal projection, in which are represented the given and required lines, *and no others*; in this respect also this construction possesses a marked advantage over the one first described.

The analysis, however, applies equally well to both; it is obvious that if a plane be drawn parallel to both lines, not only that plane but also one of the lines can be placed in a vertical position; and that, in effect, is what is done in Fig. 122.

This has been accomplished by changing the positions of the planes of projection; assuming, 1st, a new vertical plane containing CD; 2d, a new horizontal plane perpendicular to CD; and 3d, another vertical plane containing MN.

It might have been done by revolving the given lines, *first*, about a vertical axis until CD became parallel to **V**; *second*, about an axis perpendicular to **V** until CD became vertical; and *third*, about CD itself until MN became parallel to **V**. Both methods are extensively used, and that one should be adopted which is best suited to the case in hand; in the present instance, the successive rotations of the lines would have resulted in a very obscure and bewildering diagram.

94. Problem 13. *To draw a plane making given angles with the principal planes.*

Analysis. This will be best explained by the aid of Fig. 123.

Suppose the problem solved, and let tTt' represent the plane. Through any point P in the ground line draw two planes; one perpendicular to the horizontal trace, cutting the plane tTt' in the line MN, the other perpendicular to the vertical trace, intersecting tTt' in CD. These lines intersect at O, and OP is perpendicular to both. Therefore in the triangle CPD, the angle CPO is equal to the angle PDO, which is given; so that if OP be determined, the length of PC can be found, thus fixing the point C in the horizontal trace. And OP can be determined if PN is assumed; for

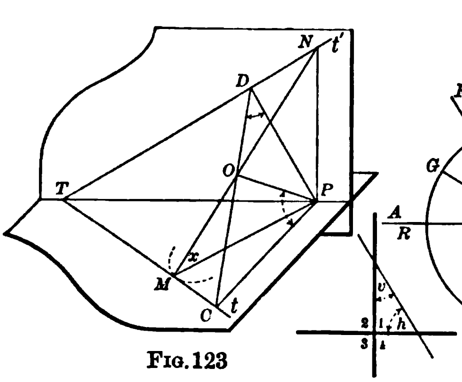


FIG. 123

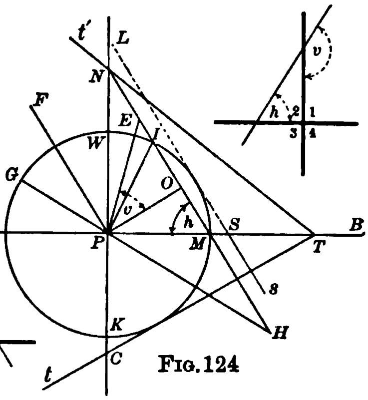


FIG. 124

since the angle at M is given, the triangle PMN can be constructed, and OP is perpendicular to MN. But PM being perpendicular to the horizontal trace, tT must be tangent to a circle described in the horizontal plane about P as a centre, with radius PM; and Tt' must pass through the assumed point N.

Construction. About any point P in **AB**, Fig. 124, describe a circle with any convenient radius PM. Through M draw a line, making the angle PMO equal to the given angle h with the horizontal plane, and cutting at N the indefinite perpendicular to **AB**, drawn through P. Draw PO perpendicular to MN; make the angle OPE equal to the given angle v with the vertical plane, and draw PE cutting MN in E. Set off on the perpendicular through P, the distance PC equal to PE, and draw, through C, tT tangent to the circle; it is the required horizontal trace, cuts **AB** in T, and TNt' is the required vertical trace.

95. Should the assigned value of v be equal to $90° - h$, the

intersection E will fall at I, C will fall at K, and the plane will be parallel to **AB**, as shown in the small profile at the left. If $v = 90°$, as OPF, the intersection is at infinity, the horizontal trace is tangent to the circle at M, and the vertical trace coincides with MN. If v is greater than 90°, as OPG, then GP produced cuts the prolongation of NM at H below **AB**, and the distance PH is to be set up as PL above **AB**; the horizontal trace is LSs tangent to the circle, and the vertical trace (not shown) will pass through the points S and N. If $v = 90° + h$, as OPR, then RP coincides with **AB**, its prolongation traverses M, and PW being equal to the radius, the horizontal trace through W, and therefore the vertical trace through N, will be parallel to **AB**, and the position of the plane is shown in the small profile at the right.

Should the assigned value of v be less than $90° - h$, or greater than $90° + h$, it is clear that the determining points of intersection with MN will fall between I and M; and since a tangent to a circle cannot be drawn through a point within the circumference, the conditions cannot be satisfied.

96. The principles and methods developed in the discussion of the preceding problems are sufficient for the solution of all others relating solely to the point, right line, and plane. The geometrical principles involved are simple enough, but thorough mastery of the manipulations can be acquired only by applying them to a great variety of cases; the examples already given illustrating the fact that slight changes in the data may cause the resulting constructions to differ widely in appearance.

A very beneficial exercise for those desirous of attaining proficiency, will be found in working the problems backward; that is to say, in assuming the points and lines of the construction to be arranged in such relations as may be desired, and then, by reversing the steps of the operation, ascertaining the conditions which would lead to that development. Indeed this inverted procedure is almost necessary in many cases to the production of a clear and explanatory illustrative diagram, and has been used in numerous instances in preparing the figures for this volume. In practical operations, of course, the conditions must be taken as they are, let the result be what it may.

For ready reference, we add here a few applications and constructions which may subsequently be found useful.

MISCELLANEOUS EXAMPLES.

97. EXAMPLE 1. *Given a right line in a given plane. To draw in that plane another line cutting the first in a given point at a given angle.*

Construction. In Fig. 125, MN is the given line, O the given point. Revolving the given plane tTt' about Tt into **H**, N falls at

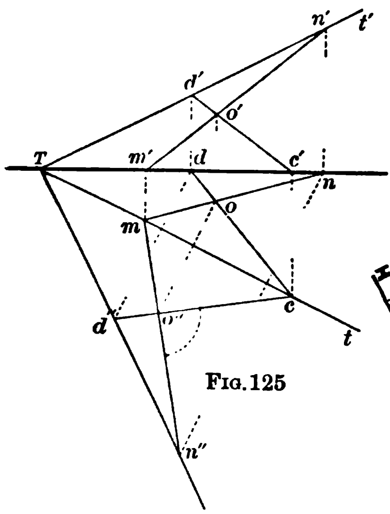


FIG. 125

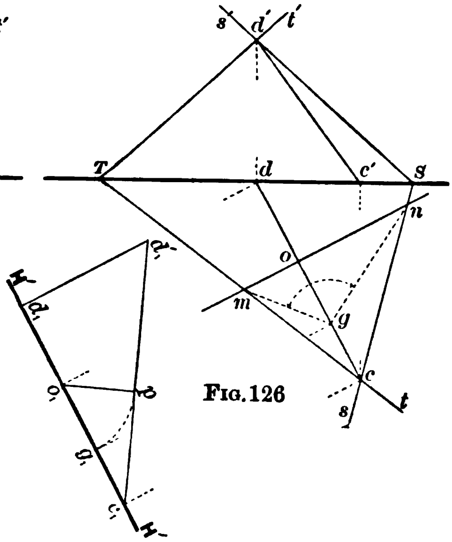


FIG. 126

n'', Tn'' being equal to Tn'; M remains fixed, and O falls at o'' on mn''. Draw $co''d''$ making with mn'' the given angle $co''n''$; it is the required line in the revolved position of the plane. In the counter-revolution, c, being in the axis, remains fixed, d'' goes to d', the distance Td' being equal to Td''; c' and d are found in **AB**, and cd, $c'd'$, are the projections of CD the line required.

98. EXAMPLE 2. *Given a right line in a given plane. To draw another plane cutting the first in the given line, and making a given angle with the given plane.*

Construction. In Fig. 126, CD is the given line in the plane tTt'. Through any convenient point o draw an indefinite perpendicular to cd, cutting Tt in m. Make a supplementary projection

on the horizontal projecting plane of CD; the horizontal plane here appears as $H'H'$, in which c_1 lies, $d_1d'_1$ is equal to dd', and CD is seen in its true length as c_1d_1'; also, o and m fall together at o_1. Draw o_1p, perpendicular to c_1d_1', and on $H'H'$ set off o_1g_1 equal to it; project g_1 back to g on cd, draw mg, and make the angle mgn equal to the given angle. Then gn produced cuts mo, produced if necessary, in n, a point in the horizontal trace required; c is a point given, and $scnS$ is the horizontal trace, $Sd's'$ being the vertical trace, of the second plane.

Note. By reference to Figs. 112 and 113, it will be perceived that this operation is nearly the converse of the process of measuring the angle between two planes. And it should now be readily understood that, instead of making a supplementary view, the horizontal projecting plane of CD might be revolved into either **H** or **V**, and after the determination of o_1g_1, revolved back to its original position.

99. EXAMPLE 3. *To draw a plane parallel to a given plane, at a given distance from it.*

Construction. In Fig. 126, let tTt' be the given plane. Draw a plane lLl' perpendicular to tT, and revolve it into **H** about its horizontal trace; cd'' is the revolved position of its intersection with the given plane. Make cg, perpendicular to cd'', equal to the

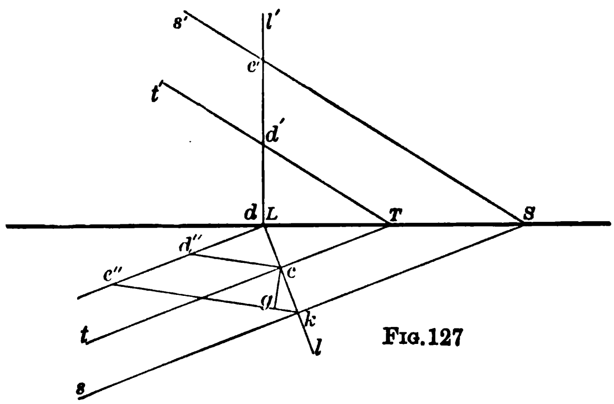


FIG. 127

given distance, and draw $e''k$ parallel to cd''; it is the revolved position of the line cut from the plane lLl by the required plane, and cuts Ll in k, a point in the horizontal trace sS; which is parallel to

tT. The vertical trace Ss' is parallel to Tt'; also, it cuts Ll' in de''.making de' equal to de .

100. Example 4. *To draw the projections of a circle in a given plane.*

Argument. Let C on the horizontal line CO in the plane tTt', Fig. 128, be the centre, about which is to be drawn a circle of a given diameter. Through c draw an indefinite perpendicular to Tt, and revolve the plane about Tt into H, where C falls at c''; draw the circle, and circumscribe it by a square, whose sides are

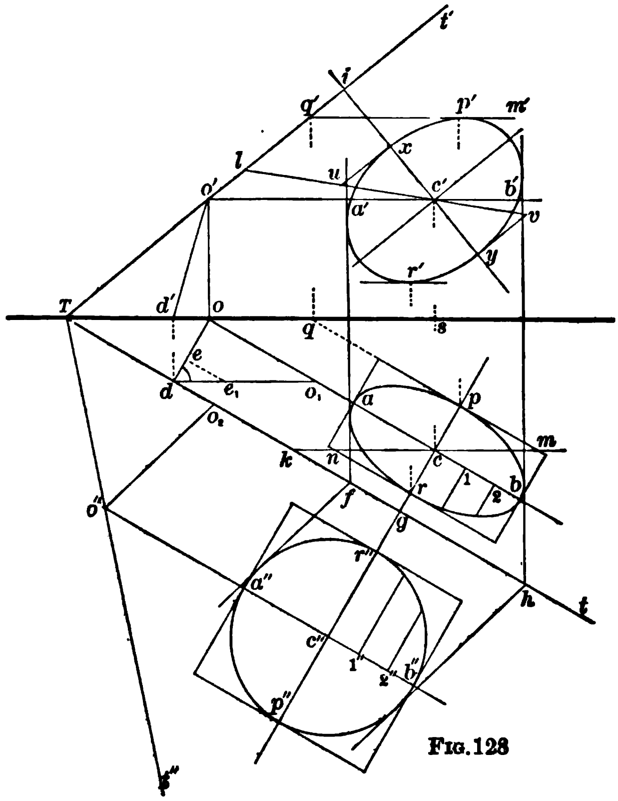


Fig. 128

parallel and perpendicular to Tt. At points on the diameter $a''b''$, as $1''$, $2''$, draw ordinates to the circle: during the counter-revolution, the lengths of $a''b''$ and of its subdivisions will remain unchanged, and the sides of the square which are parallel to the axis will remain parallel to it and to each other. So that in the projec-

tion the square will appear as a rectangle, and the circle as a curve inscribed within it; and, since the radius and the ordinates which are perpendicular to Tt are foreshortened in the same proportion, it follows that this curve is an ellipse, whose major axis ab is equal to the given diameter of the circle.

Let do and oo' be the traces of a plane, perpendicular to Tt, cutting the given plane in the line DO; revolving this line about do into **H**, it falls at do_1: on this set off $o_1e_1 = ac$, then on counter-revolution e_1 falls at e. This gives oe as the projected length of the radius; set off cr and cp equal to it, and the length rp of the minor axis of the ellipse is determined.

101. Construction. Draw through c a parallel to Tt, and on it set off ca, cb, equal to the given radius. Draw through c a perpendicular to Tt, cutting it at g; on Tt set off gk equal to sc', the distance of C from **H**; draw kc, produce it, and on it set off cn, cm, equal to the given radius. Through n and m draw parallels to Tt, thus determining rp the minor axis. It is evident that kc is parallel to do_1 of the preceding argument, because oo_1 is equal to oo', and that again to sc'.

A similar argument would apply to a square circumscribing the circle, with its sides parallel and perpendicular to the vertical trace. Therefore in the vertical projection draw through c' a parallel to that trace, and on it set off the major axis equal to the given diameter. Draw through c' a perpendicular to the trace, cutting it at i: set off il equal to sc, the distance of C from **V**: draw a line through l and c', and on it set off $c'u$, $c'v$, equal to the given radius. Through u and v draw parallels to Tt', determining x and y the extremities of the minor axis. The two ellipses may now be drawn by any of the usual methods.

102. It is obvious that p is the horizontal projection of the highest point in the curve, and mpq of the tangent to the curve at that point; the vertical projection of this tangent is $q'm'$ parallel to **AB**, and p' lies on $q'm'$: in like manner, ux, yv, being parallel to the vertical trace, will be horizontally projected as lines parallel to **AB**, which will also be tangents to the horizontal projection of the circle.

Moreover, the ellipses will be limited at the right and left by

two common tangents perpendicular to **AB**. The exact location of these can be found if desired by regarding each as the intersection of tTt by a profile plane. Draw through O a plane perpendicular to **AB**; it cuts the given plane in a line which, when revolved into **H** about Tt, takes the position o_2o''. Parallel to this draw two tangents to the circle, cutting Tt in f and h; perpendiculars to **AB** through these points are the tangents required.

103. The fact that in either projection, lines perpendicular to the trace are all equally foreshortened, while those parallel to it are not foreshortened at all, can be advantageously applied in many other cases. Referring to the figure, it is seen that in the horizontal projection the apparent length, cr, of the radius is to the actual length as the base do of the triangle doo_1 is to the hypothenuse do_1.

Now let two scales be made, whose units are to each other in this proportion, and divided into the same number of equal parts; then by measuring the ordinates, as 1″, 2″, with the larger scale, and setting them off, as 1, 2, with the smaller scale, the projection of any plane curve can be constructed, often more rapidly than by any other means.

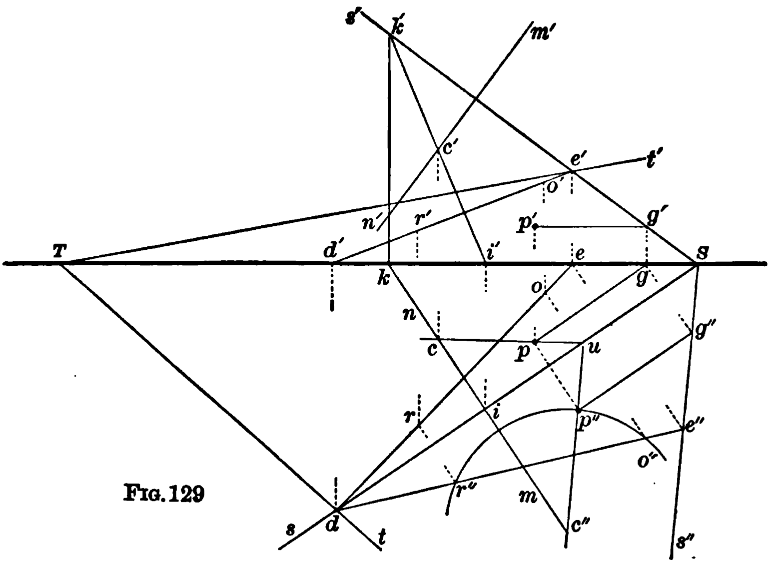


FIG. 129

104. EXAMPLE 5. *To revolve a given point about a given line*

which does not contain the point, into a given plane which contains neither.

Analysis. The plane of rotation passes through the given point, is perpendicular to the given line, and cuts it in the centre of the circular path. It also cuts the given plane in a line; and if the given point reach the given plane at all, it will be at some point of this line.

Construction. In Fig. 129, P is the given point, MN the given line, tTt' the given plane. Draw through P the plane sSs' perpendicular to MN; it cuts tTt' in the line DE, and MN in the point C. Revolving the plane sSs' about its horizontal trace into H, DE falls in the position de'', C falls at c'' and P at p''. About c'' describe a circular arc through p''; it cuts de'' in the points r'' and o''. Making the counter-revolution, c'' and p'' return to their original positions, and r'' and o'' are found respectively at R and O on the line DE: these are the required points in the given plane, at one of which P must fall when revolved about the given line into that plane.

CHAPTER III.

GENERATION AND CLASSIFICATION OF LINES AND SURFACES.—TANGENTS, NORMALS, AND ASYMPTOTES TO LINES.—OSCULATION, RECTIFICATION, RADIUS OF CURVATURE.—TANGENT, NORMAL, AND ASYMPTOTIC PLANES AND SURFACES.

GENERATION AND CLASSIFICATION OF LINES.

105. Every line may be generated by the motion of a point, which is regarded as a material particle. Any two successive positions of the generating point, having no assignable distance between them, are called **consecutive points** of the line; practically, they may be considered as coincident. But in going from one of these positions to the next, the generating point moves in a **determinate direction**, which cannot be conceived to differ from that of the right line joining those consecutive points. This infinitely short right line is called an **elementary line**: and every line may be regarded as made up of an infinite number of these rectilinear **elements**. Any two consecutive elements, since they have one point in common, must lie in one plane; but they may have different directions.

A line thus generated is called the **locus** of the successive positions of the moving point.

106. Lines are divided into classes, according to the law of the motion of the generating point, as follows: 1. RIGHT LINES. 2. SINGLE-CURVED LINES. 3. DOUBLE-CURVED LINES.

If the point moves always in the same direction, all the elements lie in the same direction, and the line is a **right line.**

If the point in moving continually changes its direction, no two consecutive elements have in general the same direction, and the **line is a curve.**

If **all the elements** of a curve lie in one plane, the line is of **single curvature.**

If **no three consecutive elements** lie in the same plane, the line is of **double curvature.**

This last may be more clearly seen by the aid of Fig. 130, where m, n, o, p, r, are the horizontal, and m', n', o', p', r', are the vertical, projections of five consecutive points, the elements MN, OP,

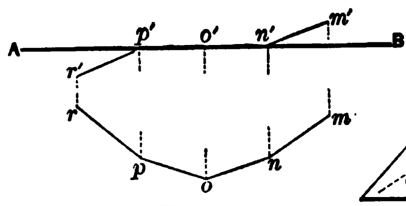


FIG. 130

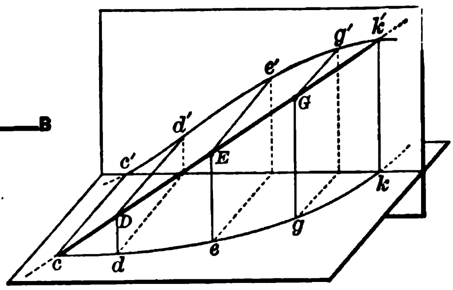


FIG. 131

etc., being enormously magnified. The three points N, O, P, and therefore the two elements NO and OP, lie in the horizontal plane, but M lies above and R lies below it. The plane of MN and NO is oblique, on being its horizontal trace, and it does not contain OP: in like manner, op is the horizontal trace of the plane determined by RP and PO, which does not contain NO. Thus it is seen that **any three**, but **no four**, consecutive points—or, what is the same thing, **any two**, but **no three**, consecutive elements—lie in the same plane.

REPRESENTATION OF CURVES.

107. A curve is represented by its projections; which contain the projections of all its points, as shown in Fig. 131. And the curve is in general fully determined if its projections on the principal planes are given. For the projections d, d', for instance, suffice to locate the point D in space (**8**), and the same is true of all the other points.

The traces of a curve are found in the same manner as those of a right line (**25**). Thus, in Fig. 131, let the horizontal projection deg be produced to cut the ground line at k; then a perpendicular to **AB** at that point, cutting the vertical projection in k', will lie in **V**, and K is the vertical trace of the curve.

No projection of a *double-curved* line can in any case be a right line. But if the plane of a *single-curved* line be perpendicular to any plane of projection, the projections of all points of the curve will lie in its trace, which is a right line. If the plane of the curve be parallel to any plane of projection, its projection on that plane will be similar and equal to the curve itself; because the projection of each element will be parallel and equal to itself (**14**). If the plane of the curve be perpendicular to the ground line, its projections on both the principal planes will be right lines perpendicular to **AB**; the curve is not undetermined, but is seen in its true form when projected on a profile plane.

108. A curve may also be generated by the motion of a right line whose successive positions intersect in a series of points. Thus, in Fig. 131*a*, the lines *L*, *M*, intersect at *m*; *M* cuts *N* at *n*,

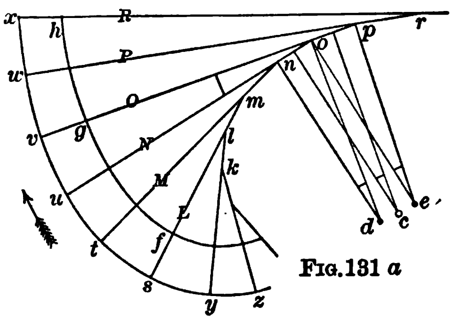


FIG. 131 *a*

and so on. Now if the points *m*, *n*, *o*, *p*, etc., are consecutive, the line joining them will be a curve, called the **envelope** of the various positions *L*, *M*, *N*, etc., of the moving line.

If in this figure the lines *L*, *M*, *N*, etc., lie in the plane of the paper, the curve will also lie in that plane. But if they be regarded as the projections of lines inclined to that plane, equally or unequally, it is clear that the envelope will be a line of double curvature. In this case, the lines *L*, *M*, determine one plane; the lines *M*, *N*, determine another, and *M* is the intersection of these two planes. Since the like is true of any three of the lines taken in order, **a double-curved line may also be generated** by the motion of a plane whose successive positions intersect in a series of lines. These lines will in general intersect each other two and two in con-

secutive points, and their envelope will be a line of double curvature.

TANGENTS, NORMALS, AND ASYMPTOTES TO LINES.—OSCULATION, RECTIFICATION, INVOLUTE AND EVOLUTE.—RADIUS OF CURVATURE.

109. The Tangent to any line, is a *right line* drawn through two of its consecutive points. These two points determine the direction of the tangent, and practically coincide in the **point of contact.**

If the given line be of single curvature, the tangent will lie in its plane; for it contains two points of the curve, and they lie in that plane. If one right line be tangent to another right line, the two lines will coincide; for they have two points in common.

Two curves are tangent to each other when they have two consecutive points in common; that is to say, when they have a common tangent at a common point.

In Fig. 131a, the lines L, M, N, etc., will be tangents to the enveloping curve when the points m, n, o, etc., become consecutive. Suppose now a series of curves, respectively tangent at these points to the same lines; then the curve $mnopr$ will be tangent to them also. And in fact the term **envelope** is used in a general sense, to designate the line tangent to any given series of lines, straight or curved.

If two lines are tangent to each other, their projections on any plane will also be tangent to each other. For the lines have two consecutive points in common, and the projections of these will be consecutive points, common to the projections of the lines.

The converse of this is not necessarily true. But if the vertical projections are tangent to each other, and also the horizontal projections, and the points of contact are the corresponding projections of a point common to the two lines, the lines themselves are tangent to each other. Because the projecting perpendiculars at the common consecutive points, will determine two consecutive points in space, which will be common to the two lines.

The angle made by a curve at any point with a plane or a right line, is the same as that made by its tangent at that point; because

the tangent contains the rectilinear element of the curve. Similarly, the angle between two curves is the same as the angle between their tangents at the common point.

110. Osculation. Osculating Plane. A right line can in general contain only two consecutive points of a curve. But two curves may have three or more consecutive points in common; in that case the contact, which is more intimate than that of simple tangency, is called **osculation.**

Through the tangent to a curve an infinite number of planes can be passed, and all these planes are tangent to the curve. Each will in general contain only two consecutive points of the curve; but one of them may be so placed as to contain *three*, and this is called the **osculating plane.** Thus, in Fig. 131a, the line M contains the point m, the line N contains the points n and o; therefore the plane determined by M and N passes through m, n, o. The osculating plane of a single-curved line is, evidently, the plane of the curve itself.

A Normal to any line is a perpendicular to its tangent, at the point of contact. An infinite number of such perpendiculars can be drawn, all of which lie in one **normal plane.** When not otherwise specified, "**the normal**" to any curve is understood to be that one which lies in the osculating plane.

The rectification of a curve means the determination of a right line equal to it in length. If the curve be so moved upon a fixed tangent that the consecutive points of the curve in their order, come into coincidence with those of the right line in their order, the motion is one of **rolling contact**; the curve measures itself off upon the tangent, and the part rolled over is equal in length to the arc of the curve which has rolled over it.

111. Involute and Evolute. If the tangent roll upon a fixed curve, any point of it will describe a second curve, which is called an **involute** of the first; and the fixed curve is called the **evolute** of the second.

In Fig. 131a, suppose a thread, wound upon the broken line knr, to be kept taut, and unwound as indicated by the arrow. Then it is clear that the path of s, a point upon the position L, will be made up of a series of circular arcs, st, tu, etc., of which the

centres are the points m, n, o, etc. When these points become consecutive, the change of centres and radii goes on continuously, the broken line becomes the **evolute**, to which the positions of the thread, L, M, N, etc., are all tangent; and the path svx becomes an **involute** of knr. But the nature of the motion is not changed; the tracing point at any instant is describing a circle about the point of contact, and therefore moving in a direction perpendicular to the radius. Consequently, the *tangent to the evolute* is always *normal to the involute*.

What is true of the motion of s is equally true of the motion of any other point f, situated upon the same line L; whose path fgh also becomes ultimately an involute of the curve knr. Hence a single-curved evolute may have an infinite number of involutes in its own plane. These are not in general *similar* curves; but they are *parallel*, in the sense that the normal distance between any two of them is constant. On the other hand, since the intersections of consecutive normals to any curve determine an evolute, a single-curved line can have but one evolute in its own plane. And this, being the envelope of **all** the normals to the given curve, whether consecutive or not, can be drawn with much precision, if the curve is such that the direction of its tangent, at any point assumed at pleasure, can be determined.

112. Centre of Curvature. In Fig. 131a, let no, op, be two consecutive elements of a curve; these, if produced, form two successive tangents N, O, whose included angle, called the **angle of contingence**, is a measure of the **rate of curvature** at o. Draw a perpendicular to each of these elements at its middle point; these perpendiculars intersect at c, the centre of a circle passing through n, o, and p.

At n and o, draw perpendiculars to N, and at o and p, perpendiculars to O; the second pair of perpendiculars will cut the first pair at the points d, e, and the three angles at d, c, e, will be equal to each other and to the angle of contingence. All these perpendiculars are understood to lie in the osculating plane. Now in the case of the actual curve here represented by a broken line, the points d and e will coincide with c; for the distances on, op, being infinitely small, the three perpendiculars to N will form one

single line; and the three perpendiculars to O will form another. The angle of contingence is also infinitely small; nevertheless the point c, then the intersection of two consecutive normals, is in general at a finite distance from o, and is the centre of a circle whose circumference contains the three consecutive points n, o, p, of the curve.

This, then, is the **osculating circle**; and since its circumference has the same rate of curvature as that of the curve itself at o, its centre c is called the **centre of curvature**, and co is called the **radius of curvature**, of the given curve at that point.

The intersections of successive normals to any curve will be a series of consecutive points, determining in general another curve, which is the locus of the centre of curvature. In the case of a *plane curve*, this locus will be the *evolute* of the given curve. Thus, in Fig. 131a, N and O, consecutive normals to svx, meet at o on the curve knr, and o is the centre of curvature at v. This does not hold true of double-curved lines.

113. **An Asymptote** to a line is another line which the given line, during a portion of its course, continually approaches, becoming tangent to it only when its own length becomes infinite.

Ordinarily, the asymptote to a plane curve is a right line, which becomes tangent to the curve at an infinite distance. But this is not essential: for example, the evolute of the Archimedean spiral has a circular asymptote of finite diameter, and the curve lies wholly within the circumference; again, two conjugate hyperbolas, having common rectilinear asymptotes, are asymptotic to each other. Also, a right line may be an asymptote to a line of double curvature.

GENERATION AND CLASSIFICATION OF SURFACES.

114. **Every surface may be generated by the motion of a line.** This moving line is called the **generatrix**, and its different positions are called **elements**, of the surface. Any two successive positions of the generatrix, having no assignable distance between them, are called **consecutive elements**; practically they may be regarded as coinciding.

Surfaces may be separated into two grand divisions, according to the form of the generatrix, viz.:

I. Ruled Surfaces, which contain rectilinear elements.

II. Double-Curved Surfaces, which have no rectilinear elements.

In other words, the surfaces of the first division *can* be generated by the motion of right lines, while those of the second division *cannot*. The former **may** also be generated by the motion of curved lines; the latter **cannot** be generated without it.

115. A right line may move so that all its positions lie in the same plane. It may move otherwise; in which case any two consecutive positions either will lie in the same plane, or they will not. According to the law of the motion of the **rectilinear generatrix**, then, ruled surfaces are subdivided into three classes, as follows:

1. Plane Surfaces........**All** the rectilinear elements lie in the same plane.

2. Single-curved Surfaces..**Any two** consecutive rectilinear elements lie in the same plane.

3. Warped Surfaces.......**No two** consecutive rectilinear elements lie in the same plane.

PLANE SURFACES.

116. The **Plane Surface** is unique. That is to say, there is but one form of plane, and there neither are nor can be any different kinds. All planes are flat, and one is no flatter than another.

The rectilinear generatrix may move so as to touch another right line, remaining always parallel to its first position; so as to touch two other right lines which are parallel to each other, or which intersect; or it may revolve about another right line to which it is perpendicular.

Acquaintance with the nature and properties of planes was necessarily assumed at the outset; the methods of representing them, and of assuming points and lines in them, have already been described.

SURFACES OF SINGLE CURVATURE.

117. Single-curved surfaces are of three varieties, viz.:

1. CONES......In which all the rectilinear elements intersect in a common point.

2. CYLINDERS..In which all the rectilinear elements are parallel to each other.

3. CONVOLUTES..In which the consecutive elements intersect **two and two**, no three having a common point.

CONICAL SURFACES.

118. In generating a **cone**, the right line moves so as always to touch a given curve called the **directrix**, and also to traverse a given point called the **vertex**. Since the generatrix is indefinite in length, the surface is divided at the vertex into two parts, called respectively the **upper and lower nappes**. It is clear that in the case of a given cone, any line drawn upon the surface so as to cut all the elements may be taken as a **directrix**, and any element as the **generatrix**.

The cone may also be generated by the motion of a curve which always touches a given right line, and changes its size according to a proper law.

119. The portion of the cone usually considered, is included between the vertex and a plane which cuts all the elements; the curve of intersection is called the **base**, and its form gives a distinguishing name to the surface—as a cone with a circular, a parabolic, an elliptical, or a spiral base, as the case may be. If the base has a centre, a right line drawn through this centre and the vertex is called the **axis** of the cone. The point in which any element pierces the plane of the base is called the **foot** of the element.

A definite portion of either nappe, included between two parallel planes which cut all the elements, is called a **frustum** of the cone, the limiting curves being called respectively the **upper and lower bases**.

A secant plane through the vertex cuts the cone in rectilinear elements intersecting the base.

120. A right cone is one all of whose rectilinear elements make equal angles with a right line passing through the vertex, which is called the **axis.** This is also called a **cone of revolution,** since it can be generated by revolving the hypothenuse of a right-angled triangle about one of its sides as an axis.

If the directrix of a cone be changed to a right line, or if the vertex be placed in the plane of a single-curved directrix, the cone will degenerate into a plane.

If the vertex be removed to an infinite distance, the rectilinear elements will be parallel to each other, and the cone will become a cylinder.

121. Representation of the cone. A cone is represented by the projections of the vertex, one of the curves of the surface (usually its plane base), and the principal rectilinear elements. Thus, in Fig. 132, let O be the vertex; draw $mxny$, the horizontal projec-

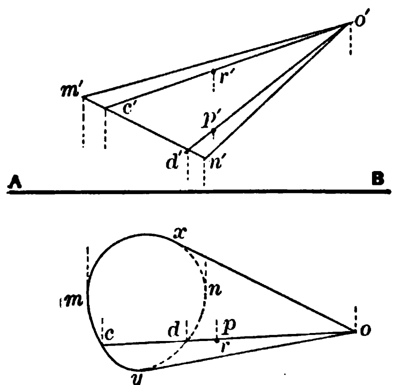


Fig.132

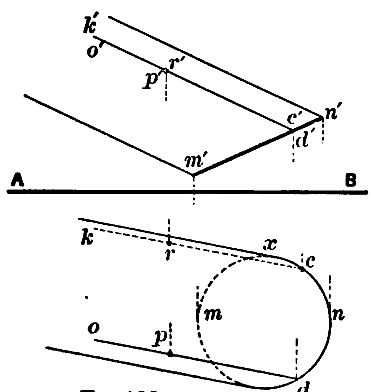


Fig.133

tion of the base, and ox, oy, tangent to that curve: this completes the horizontal projection of the cone. The plane of the base in this case is perpendicular to **V**; its vertical projection is therefore a right line, limited at m' and n' by tangents to the horizontal projection perpendicular to **AB**; the vertical projections $m'o'$, $n'o'$, of the extreme visible elements, complete the representation of the surface.

To assume a rectilinear element, assume a point on the curve of the base, as C or D, and draw through it a right line to the vertex.

To assume a point on the surface, assume one of its projections, say the horizontal, as p; through p draw the horizontal projection of an element, op; this element intersects the base at D, and p' must lie on $d'o'$, the vertical projection of the element. The horizontal projecting plane of DO cuts the cone in another element, CO, having the same horizontal projection; upon this lies another point R, whose horizontal projection r coincides with p.

122. Particular attention is called to the fact that the cone is here shown with its base *not* situated in the horizontal plane. And it is so shown for the purpose of illustrating and emphasizing another fact, viz., that the **projection** of the base must not be confounded with **the base itself**. It is very natural and proper to place the base of a cone or a cylinder upon the horizontal plane; but if it be always so placed, experience has proved that the above necessary distinction is very apt to be lost sight of, which may lead to serious errors in the subsequent applications of problems relating to these surfaces.

CYLINDRICAL SURFACES.

123. The Cylinder, as intimated in (**120**), is merely that limiting form of the cone in which the vertex is infinitely remote; and it may be generated by a right line which moves so as always to touch a given curve, and have all its positions parallel. In the case of a given cylinder, any line of the surface which cuts all the rectilinear elements may be taken as a directrix, and any one of those elements as the generatrix.

A cylinder may also be generated by a curvilinear generatrix, all of whose points move in the same direction and with the same velocity.

124. A plane cutting all the rectilinear elements, intersects the cylinder in a curve called its **base**; whose form, as in the case of the cone, gives a distinguishing name to the cylinder. If the base has a centre, a right line drawn through the centre, parallel to the elements, is called the **axis**.

If a definite portion of the surface included between two parallel planes is considered, the two curves of intersection are called the **upper and lower bases**.

A plane parallel to the rectilinear generatrix cuts the cylinder, if at all, in rectilinear elements intersecting the base.

125. **A right cylinder** is one whose rectilinear elements are perpendicular to the plane of the base; and the base itself is then said to be a **right section.**

A right cylinder with a circular base is also called a **cylinder of revolution**, since it may be generated by revolving one side of a rectangle about the opposite side as an axis.

If the curvilinear directrix of any cylinder be changed to a right line, the surface will degenerate into a plane.

126. The projecting lines of the various points of a curve, as seen in Fig. 131, are rectilinear elements of a right cylinder, whose base, in the plane of projection, is the projection of the curve. Thus the curve in space is determined by the intersection of two cylinders, called respectively the **horizontal and vertical projecting cylinders.**

127. **The representation of the cylinder** differs from that of the cone only in this, that the projections of the rectilinear elements are parallel instead of convergent. If a limited portion is to be represented, the projections of both bases must be drawn; if not, the projections of the extreme visible elements may terminate indefinitely, as shown in Fig. 133.

To assume a rectilinear element, assume a point on the curve of the base, as C or D, and draw through it a right line parallel to the rectilinear generatrix.

To assume a point on the surface, assume one of its projections, say the vertical, as p'; through this draw the corresponding projection $o'p'$ of an element; this element intersects the base at D, and p must lie on do, the horizontal projection of the element. The vertical projecting plane of DO cuts the cylinder in another element, CK, having the same vertical projection; and upon this element lies another point, R, of the surface, whose vertical projection coincides with that of P.

CONVOLUTE SURFACES.

128. **The Convolute** may be generated by a right line which moves so as always to be tangent to a line of double curvature. Any

two consecutive rectilinear elements, but no three, will lie in the same plane; for they are the extensions of the elements of the **directrix**, of which (**107**) any two, but no three, consecutive ones intersect each other.

Suppose a piece of paper cut in the form of a right-angled triangle to be wrapped around a regular polygonal prism, Fig. 134, its base becoming the perimeter of the base of the prism. Then the hypothenuse will become a broken line, each portion lying in a

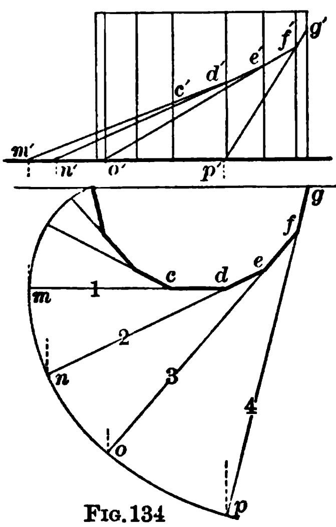


FIG. 134

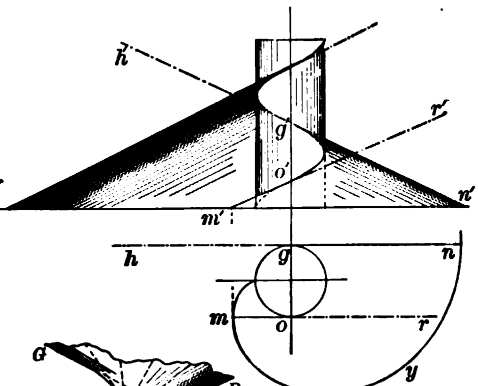


FIG. 135

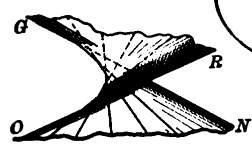


FIG. 136

face of the prism and being equally inclined to its edges. In unwinding, the paper turns about each edge in succession as upon a hinge, until it coincides with the plane of the next face; when the free portion of the hypothenuse will be an extension of, and therefore tangent to, the element of the broken line which lies in that face. And, considering the successive positions of the hypothenuse, it is seen that No. 1 intersects No. 2 at *D*, No. 2 intersects No. 3 at *E*, No. 3 cuts No. 4 at *F*, and so on; but No. 1 does not intersect No. 3, nor does No. 2 intersect No. 4. Now let the sides of the polygon be indefinitely increased in number; the base will ultimately become a circle, the prism will become a cylinder; the broken line will become a helix, and its tangents, then consecutive,

will lie in a continuous surface, of which a limited portion is shown in Fig. 135.

The point M of the unwinding paper will always lie in the curve myn in the horizontal plane; this curve, which is the involute of the circular base, is the horizontal trace of the surface, which, expanding as it rises, winds around the cylinder in convolutions like those of a sea-shell. The cylinder itself has no connection with the surface; it is introduced merely in order to throw the nearer portion of the convolute into stronger relief.

As in the cases of the cone and the cylinder, the curve of intersection with any plane which cuts all the rectilinear elements may be taken as the base of the convolute.

129. The Edge of Regression. The generatrix, in Fig. 135, is shown as limited in length; thus, the elements MO, NG, terminate at O and G, their points of contact with the directrix.

But they may be continued past those points, as indicated in dotted lines; and their extensions, OR and GH, lie in a continuation of the surface, which expands in other successive whorls: and these two portions, or nappes, of the convolute have in common the helical directrix.

But it is to be noted that this is **not a curve of intersection**; for, as shown in Fig. 136, which represents a detached portion of the surface on a larger scale, the rectilinear elements which lie upon the lower nappe neither pierce the upper nappe nor cut the helix. Indeed, they are tangent to the helix by hypothesis; and the surface is continuous and unbroken, although reflected sharply upon itself, and forming at the helix what is called an **edge of regression.** This is a limiting line, at which a surface terminates abruptly by the law of its generation: it is always formed by the intersection of consecutive generating lines, whether they are right lines, as in this case, or curved ones.

A surface may also be reflected, or bent back, in an analogous manner, along a line which is not thus formed; the limiting line in that case is called a **gorge line.**

Since there is an infinite number of double-curved lines, a great variety of convolutes may also exist, with peculiarities depending upon those of their directrices; but the one above described will

suffice for illustration. It has been selected for the reasons that it is not only as simple as any, but possesses some interesting properties, which will be noticed in due course, and render it practically more important than others. The methods of constructing and representing it will also be subsequently discussed in connection with problems relating to it.

GENERATION OF SINGLE-CURVED SURFACES BY MOVING PLANES.

130. Observing that any two consecutive elements of the convolute determine an osculating plane of the directrix (**110**), and that each element is the intersection of two successive positions of the osculating plane (**108**), it will now be seen that this surface may be generated by the motion of a plane subject to the condition that it shall always be osculatory to a given line of double curvature. Since three points not in one right line suffice to locate a plane, this single condition will in general control absolutely the motion of the **plane generatrix**, and determine the form of the resulting surface. **An exceptional case occurs** when the directrix is reduced to a point, the motion of the plane being then indeterminate unless governed by another condition, which may be deduced as follows:

If, on any single-curved surface, any curve be drawn which cuts all the rectilinear elements, any two consecutive ones will intercept an element of that curve; the plane of those two elements will therefore be tangent to the curve (**110**). In the case just mentioned the surface becomes a **cone**; which may, consequently, be generated by the motion of a plane which always passes through a given point and also **remains tangent to a given curve.** If the vertex be infinitely remote, the cone will become a cylinder; all of whose elements, being parallel, are perpendicular to one and the same plane. Hence a cylinder may be generated by the motion of a plane which is always tangent to a given curve and also always perpendicular to a given plane.

WARPED SURFACES.

131. The absolute motion of a right line in space is fully determined when the simultaneous motions of any two of its points are

given in **direction and velocity**; the **form of the surface** generated by the moving line will be determined if the directions and **relative** velocities of these two motions be known.

These directions may be determined by requiring that the rectilinear generatrix shall always touch two other given lines, either straight or curved; but some third condition is necessary in order to establish a definite ratio between the velocities.

By whatever means this is accomplished, it is clear that if upon the resulting surface any other line be drawn which cuts all the rectilinear elements, that line may be taken as a third directrix; and the same surface will be produced if the rectilinear generatrix move so as always to touch this last line and the two at first assumed.

Any ruled surface whatever, then, may be generated by a right line so moving as always to touch **three given linear directrices.**

132. Cone Directer and **Plane Directer.** Suppose the three directrices to be so chosen that the resulting ruled surface is neither plane nor single-curved; then through any given point in space let a series of consecutive right lines be drawn, parallel in their order to the consecutive rectilinear elements of the given warped surface in their order. These will be elements of a cone, called the **cone directer** of the surface: and it is clear that if **any two** lines which cut all the rectilinear elements be taken as directrices, a right line moving so as always to touch those lines and have its consecutive positions parallel to the consecutive elements of the cone directer, will re-generate the same surface.

It is easy to see that the warped surface may be such that the series of lines, drawn through the assumed point parallel to its elements, shall lie in one plane—which is a limiting form of the cone; and this plane is called the **plane directer.**

Any warped surface whatever, therefore, may be generated by a right line moving so as always to touch **two given lines**, and have its consecutive positions parallel either to a given plane directer, or to the consecutive elements of a given cone directer.

133. Since every warped surface is curved, it is possible also to conceive it as being generated by a curve, which moves, and at the same time changes its form, according to some definite law.

There is a great variety of warped surfaces, with peculiarities depending on the laws of their formation. The methods of representing them, of assuming points and lines upon them, etc., require for ready apprehension a familiarity with some matters not yet discussed, and are accordingly reserved for subsequent consideration.

DOUBLE-CURVED SURFACES.

134. A Double-curved Surface is one which contains no rectilinear elements, and can be generated **only** by a curve which moves in such a manner as not to generate a surface of either of the preceding classes.

Double-curved surfaces may be either **double convex**, that is, convex in all directions, as the surface of a sphere or an egg; or **concavo-convex**, that is, convex in some directions but concave in others, as the surface of a bell or of the groove in a pulley. Both these peculiarities may exist in a single unbroken surface, as in the case of a cylindrical ring, or annular torus.

SURFACES OF REVOLUTION.

135. A Surface of Revolution is one which may be generated by the revolution of a given line about a right line as an axis.

The intersection of such a surface by a plane perpendicular to the axis is, therefore, the circumference of a circle. Consequently the surface may also be generated by a circle which, moving with its centre in the axis and its plane perpendicular to it, changes its radius according to a definite law.

This second mode of generation is the one to which the greater practical interest attaches; since it is in this manner that such surfaces, of extensive application in the mechanic arts, are actually produced in the lathe.

A plane traversing the axis is called a **meridian plane**, and its intersection with the surface is called a **meridian line**: all meridian lines of the same surface are obviously identical, and any one of them may be taken as the revolving generatrix.

136. If the revolving line be straight, it either will lie in the same plane with the axis, or it will not. If it does, it will either

be parallel to it, or intersect it; in the former case the surface will be a cylinder, in the latter a cone; and these are **the only single-curved surfaces of revolution.**

If the revolving right line does not lie in the same plane with the axis, then, **1**: Every point of it moves, therefore the consecutive positions do not intersect; and, **2**: Every point more remote from the axis moves faster than one nearer to it, therefore the consecutive positions are not parallel. The surface must, then, be warped; its meridian line, as will subsequently appear, is an hyperbola: and this is **the only warped surface of revolution.** It may also be generated by revolving an hyperbola about its conjugate axis; and the surface being unbroken, it is also known as *the hyperboloid of revolution of one nappe.*

With the exception of the three just considered, all surfaces of revolution are of double curvature.

137. If two surfaces of revolution, having a common axis, cut or touch each other at any point, they will do so all around the circumference of the circle described by that point. Thus in Fig. 137, the meridian lines *macn*, *oacp*, are tangent to each other at *a*,

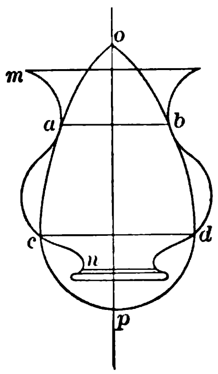


FIG. 137

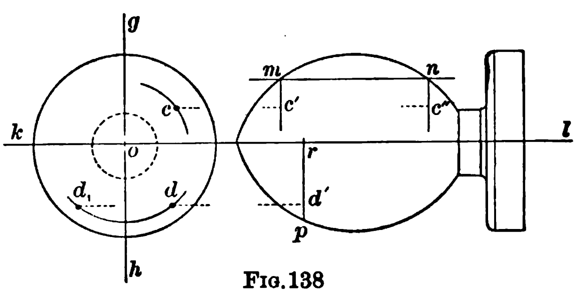


FIG. 138

and intersect each other at *c*; they will maintain these relations throughout the revolution, and the circles *ab*, *cd*, will be common to the two surfaces.

138. Representation of Surfaces of Revolution. These surfaces are represented by drawing two views, viz.: 1, **a side view**, showing the meridian contour, and should this be a broken line, such

circles as are described by the intersections; and 2, an **end view**, in which are drawn the largest circle of the surface, and such others as may be necessary in order that the drawings may be clear and easily read.

For all ordinary practical purposes, and for most of the purposes of descriptive geometry as well, any reference to a ground line is useless, if not worse. As shown in Fig. 138, a fine, continuous **centre line**, *kl*, containing the axis, should be drawn through and *beyond* both views; another one, *gh*, should be drawn at right angles to the first, through the centre of the end view: since these lines are imaginary, they should never terminate in any outline, lest they be supposed to represent lines actually existing on the surface.

The axis, when there is no reason to the contrary, is supposed to be parallel to the paper, and either horizontal or vertical as may be more convenient; in the latter case the end view is a "horizontal projection"—but whether placed above or below the side view, this horizontal projection is invariably a **top view**, and represents the object as seen from *above*, never as seen from below.

139. To assume a point on the surface, assume one of its projections, for instance c in the end view. Then the point must lie on the surface of a cylinder whose radius oc is known; draw at that distance from the axis a parallel to it, in the side view; this is the outline of the cylinder, and cuts the meridian line in m and n; these points describe circles, to one or both of which c is projected, as at c' or c''.

If the projection in the side view be assumed, as at d', then the point lies on a circle whose radius pr can be found. With this radius describe an arc about o in the end view; the other projection must lie on this arc, as at d or d_1.

TANGENT, NORMAL, AND ASYMPTOTIC PLANES AND SURFACES.

140. If on any surface, any number of lines be drawn through a given point, then the tangents to all these lines at the common point will in general lie in one and the same plane.

Such a plane is said to be **tangent** to the surface; and the point is called the **point of contact**.

If the surface is a single-curved one, no plane can be tangent to it at any point, which does not coincide with some position of the **plane generatrix.** That generatrix *contains two consecutive rectilinear elements*, and is tangent (**130**) to every curve of the surface which cuts them both. An infinite number of such curves can be drawn through any point of either; hence **the plane is tangent all along the line** in which those two elements practically coincide.

In the case of a double-convex surface, the demonstration is as follows: Let M, Fig. 138a, be a section of the surface by any

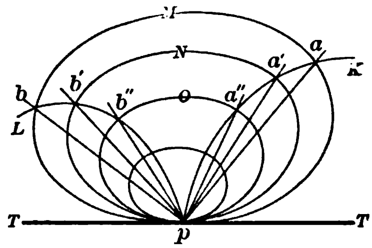


FIG. 138 a

plane, and T a tangent to it at any point p; if the plane be revolved about T as an axis, this line will also be tangent to any successive section, as N or O: for it contains two consecutive points of M, and they remain fixed during the revolution. Through p draw on the surface any other curves, as K, L, cutting M at a, b. The section N cuts these curves at a', b'; the section O cuts them at a'', b''; therefore the secants pa, pb; pa', pb'; pa'', pb'', always lie in the revolving plane. Ultimately, the section of the surface will cut K and L at points consecutive to p, and these secants will become tangents, still lying in the same plane through T. If the surface is concavo-convex, whether warped or double-curved, the same argument applies, although the forms of the sections M, N, etc., will be different.

To pass a plane tangent to any surface at a given point, therefore: Draw through the point any two intersecting lines of the surface, and at the point draw a tangent to each line; the plane determined by these two tangents is the plane required.

141. A plane tangent to any ruled surface must in general

contain the rectilinear elements which pass through the point of contact; for each rectilinear element is its own tangent (**109**), and therefore lies in the tangent plane by the preceding definition. The vertex of a cone is an exceptional case; through that point an infinite number of planes tangent to the surface may be passed, each of which contains two consecutive rectilinear elements, but no others.

Since a plane tangent to any single-curved surface is tangent all along a rectilinear element (**140**), it follows that **if** the base of the surface lie in any plane of projection, the corresponding trace of the tangent plane will be tangent to the base, at the point in which the **element of contact** pierces that plane of projection. And in any case, the right line cut from the plane of the base by the tangent plane will be tangent to the curve of the base.

142. A plane tangent to a warped surface contains the rectilinear element which passes through the point of contact; but since it does not contain the consecutive one, it cannot **in general** be tangent along the element; but there are some cases in which it is. If the surface has two sets of rectilinear elements, the tangent plane will contain both those which pass through the point of contact.

A plane containing one rectilinear element of a warped surface, **and not parallel to the consecutive ones**, will cut each of them in a point. The curve joining these points will cut the given element, and the given plane will be **tangent to the surface** at the point of intersection; for it contains the given element, which is its own tangent, and also the tangent to the curve of intersection at the point mentioned. If the given plane be parallel to a plane directer of the surface, there will be no such curve, and the plane will not in general be tangent to the surface.

Consequently, unless the projecting planes of the rectilinear elements are parallel to a plane directer, each of them will be tangent to the surface at some point. The projecting lines of these points form the projecting cylinder of the surface; and the projections of the elements, being the traces of the projecting planes, will all be tangent to the base of this cylinder, which lies in the plane of projection.

143. A plane tangent to a surface of revolution is perpendicular to the meridian plane passing through the point of contact. Because, it contains the tangent to the circle of the surface at that point; and this tangent, lying in a plane perpendicular to the axis, is perpendicular to the radius at its extremity, and also to any line joining that extremity with the axis; and both this radius and that line lie in the meridian plane.

144. Two curved surfaces are tangent to each other when they have, at a common point, a common tangent plane. Evidently, the sections of the two surfaces made by any one plane passing through the point of contact will be tangent to each other at that point.

If two surfaces of revolution having a common axis are tangent to each other at a point, they will be tangent all round the circumference of the circle described by that point in the generation of the surfaces (**137**).

If two single-curved surfaces are tangent to each other at a point of a common rectilinear element, they will be tangent **all along that element.** Because the plane tangent to either surface at any point is tangent to it all along the rectilinear element passing through that point (**140**).

It is possible also to construct two warped surfaces which shall be tangent to each other all along a common rectilinear element. They must then have, at every point of that element, a common tangent plane; the methods by which this condition can be satisfied will be explained farther on.

145. Normal Lines, Planes, and Surfaces. A right line is **normal** to a surface at any point when it is perpendicular to the tangent plane at that point.

A curved line is normal to a surface at any point when its tangent at that point is normal to the surface. When not otherwise stated, "the normal" to a surface is understood to be rectilinear.

A plane is normal to a surface at any point when it is perpendicular to the tangent plane at that point. Thus there may be an infinite number of planes normal to the surface at a point, while there can be but one normal right line, common to all these planes.

If at the consecutive points of any line upon a given surface a

series of normals to that surface be erected, they will be elements of a ruled surface **normal to the given surface.** If these normals are tangent at those consecutive points to lines lying upon any other surface, then that surface is also normal to the given surface. This relation is mutual, and the two surfaces are said to be **normal to each other.**

146. Asymptotic Planes and Surfaces. The relation between asymptotic surfaces is analogous to that between asymptotic lines. Thus if an hyperbola be made the base of a right cylinder, the plane containing its asymptote and parallel to the elements of the cylinder will be asymptotic to the surface. Again, if two conjugate hyperbolas, together with their common asymptotes, be revolved about either axis, the two hyperboloids of revolution will be asymptotic to each other and to the cone generated by the asymptotes.

CHAPTER IV.

ON THE DETERMINATION OF PLANES TANGENT TO SURFACES OF SINGLE AND OF DOUBLE CURVATURE.

PLANES TANGENT TO SINGLE-CURVED SURFACES.

147. In the construction of these problems, and of many others, there is frequent occasion to draw tangents to curves of unknown properties. A sufficient degree of accuracy for present purposes, and indeed for most practical purposes, may be obtained by means of approximating circular arcs, as follows: Let it be required to draw a tangent to the curve DE, Fig. 139, at the point P. By trial and error a centre C and radius CP can be found, such that the circular arc described about C will sensibly coincide with the given curve for a short distance on each side of the given point; the required tangent is then drawn, perpendicular to CP.

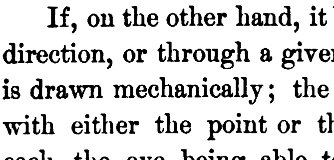


FIG. 139

If, on the other hand, it be required to draw a tangent in a given direction, or through a given point, as O: In this case the tangent is drawn mechanically; the ruler being set, not so as to *coincide* with either the point or the curve, but at a small distance from each, the eye being able to judge with perfect precision as to the equality of these distances. The point of contact is then determined by dropping a perpendicular upon the tangent from the centre C, found as above.

147a. The following constructions are more laborious, but give more precise determinations:

1. In Fig. 139a, to draw a tangent to the curve KL at any point p. Through p draw chords from any points of the curve, produce them all in one direction (say to the left), then with p as

a centre and any convenient radius draw a circular arc UV intersecting them. From this arc set off on the prolongation of each chord a distance equal to the chord itself, the chords on the left of p being set off to the left of the arc, and *vice versa;* thus $ef = pb$,

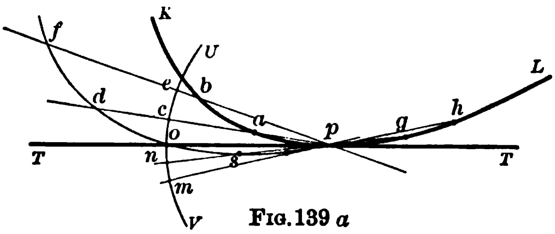


FIG. 139 *a*

$cd = pa$, $ns = pg$, and so on. The curve fds thus determined cuts UV in a point o of the required tangent TT.

2. In Fig. 139*b*, to find the point of contact, TT being tangent to the curve KL. Draw any number of chords parallel to TT, through their extremities draw parallel ordinates in opposite directions, and on each ordinate set off from TT a distance equal to the corresponding chord; as, for example, $rs = mn = gh$, $bd = ef = ac$, etc.; the curve sdn, passing through the points just located, cuts TT in the required point.

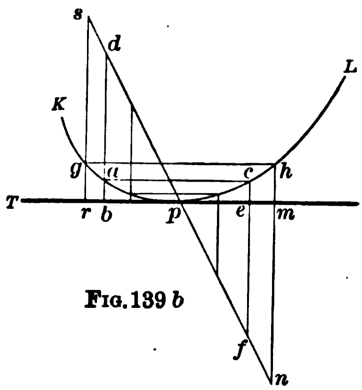


FIG. 139 *b*

Such curves are known as "curves of error," and with due care give very accurate results.

148. PROBLEM 1. *To draw a plane tangent to any single-curved surface through any given point of the surface.*

Analysis. Through the given point draw a rectilinear element, and through the foot of that element draw a tangent to the base. The plane of these two lines is the required tangent plane.

Construction. In Fig. 140, the given surface is a cone whose vertex is O, the plane of the base being perpendicular to **V**; this is represented, and the point P upon it assumed, as in Fig. 132. Through P draw the element OPC, and through its foot C draw a

tangent to the base. The horizontal projection kc of this tangent will be tangent at c to the horizontal projection of the base, and its vertical projection $k'c'$ will coincide with the vertical projection of the base. The traces of the tangent are M and G; those of the element are N and D; therefore m' and n' are points in the vertical trace, and d and g are points in the horizontal trace, of the required plane; and these traces must meet at T in the ground line.

In Fig. 141, the surface is a cylinder, with its base in the vertical plane. The tangent to the base is therefore the required ver-

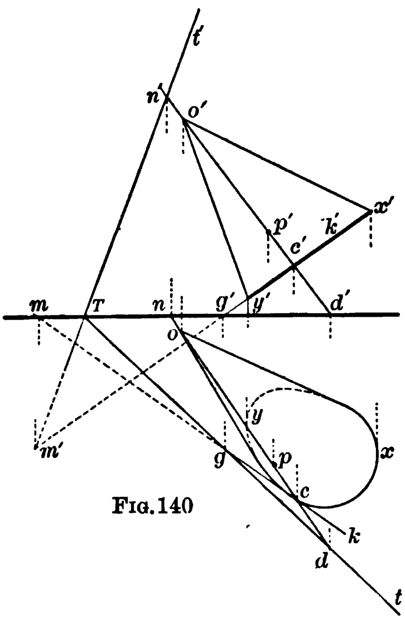


FIG. 140

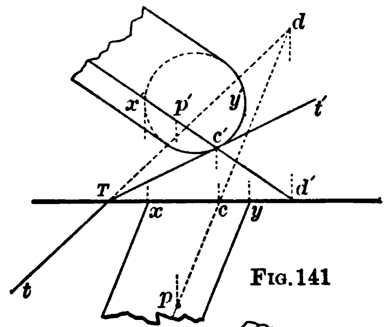


FIG. 141

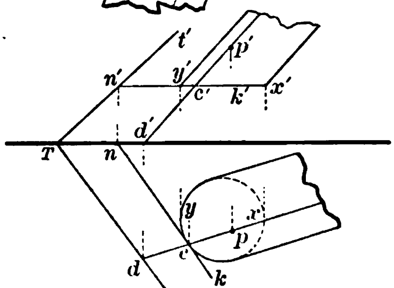


FIG. 142

tical trace, which cuts AB at T. The element PC pierces H in D, thus determining dTt the horizontal trace.

In Fig. 142, the base of the cylinder is horizontal; the horizontal trace is therefore parallel to kc, the horizontal projection of the tangent to the base. The point d, found as before, fixes the location of this trace, which, when produced, cuts AB in T. The tangent KC pierces V in the point N; and since the vertical trace of the element PC is inaccessible, the direction of Tt' is determined by drawing it through n'.

149. PROBLEM 2. *To draw a plane tangent to a cone through a given point without the surface.*

First Method. **Analysis.** Draw a line through the vertex of the cone and the given point. Through the point in which this line pierces the plane of the cone's base, draw a tangent to the base. The plane of these two lines is the required tangent plane.

Construction. Let O–XY, Fig. 143, be the given cone, P the given point. The line OP, through the point and the vertex, pierces **H** in D, and the plane of the cone's base in S. The verti-

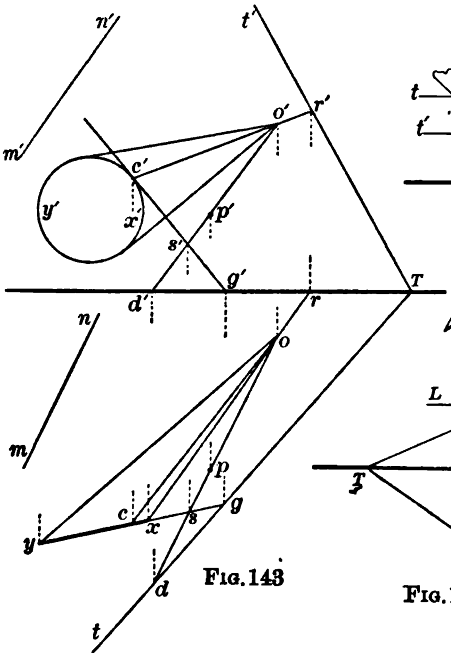


FIG. 143

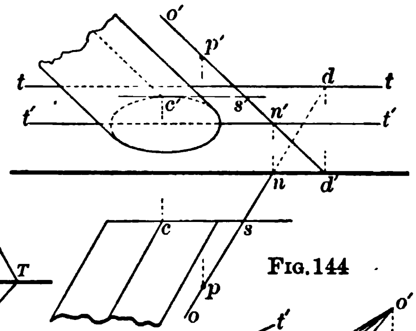


FIG. 144

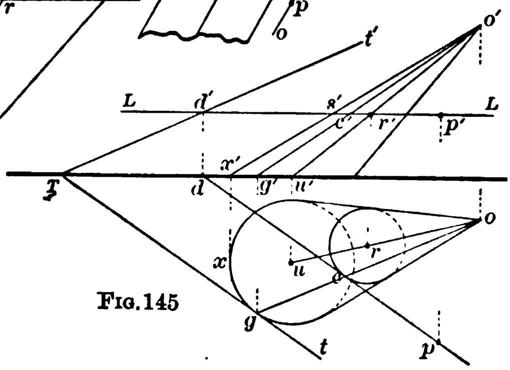


FIG. 145

cal projection of the tangent is $s'c'$ tangent to the vertical projection of the base; its horizontal projection coincides with that of the base itself, whose plane is perpendicular to **H**. This tangent pierces **H** in G; and d and g are two points in the horizontal trace of the required plane, which cuts **AB** in T, one point of the vertical trace. Another point might be determined by finding the vertical trace of OP; but in this instance a third line, the element of contact OC, has been employed instead; it pierces **V** in R, and $Tr't'$ is the required vertical trace.

When the cone becomes a cylinder, as in Fig. 144, the line through the vertex becomes parallel to the elements. In the diagram, the plane of the base is parallel to **V**, and OP pierces it at S; in this particular case the vertical projection $s'c'$ of the tangent to the base happens to be parallel to **AB**, and since SC is therefore parallel to the horizontal trace, the required plane will be parallel to the ground line; consequently it is sufficient to determine one point in each trace.

When more than one tangent to the base can be drawn, there will be more than one solution.

150. *Second Method.* **Analysis.** Pass through the given point any plane cutting all the rectilinear elements of the given surface, and in this plane draw through the point a tangent to the curve of intersection. Draw through the point of contact a rectilinear element; this element, and the tangent to the curve, determine the required plane.

Note.—This process may be applied to any single curved surface; but is of more particular advantage in cases analogous to the one herewith illustrated.

Construction. In Fig. 145 the surface is a cone, with a circular base situated in the horizontal plane: every section of it by a horizontal plane will therefore be a circle whose centre lies upon the line OU, drawn from the vertex to the centre of the base. Through the given point P draw a horizontal plane LL; it cuts OU in R, and also cuts the element OX in a point whose vertical projection is s'; therefore $r's'$ is the radius of the circular section, which moreover must be tangent to the extreme visible elements in the horizontal projection. Draw through P a tangent to this section; in this case this tangent is parallel to the horizontal trace, which is tangent to the base of the cone; so that it is not *necessary* to make use of the element of contact, CG. But, especially if the given point P is but a small distance above **H**, it would be advisable to use the vertical trace of that element if accessible, since the direction of Tt' would be thus more accurately determined.

151. PROBLEM 3. *To draw a plane tangent to a given cone and parallel to a given right line.*

Analysis. Through the vertex of the cone draw a parallel to the

given line; and from the point in which it pierces the plane of the cone's base, draw a tangent to the base. The plane of this tangent and the parallel, is the tangent plane required.

Construction. If in Fig. 143 we suppose the line OP to be determined by the condition that it shall be parallel to a given line MN, the construction is the same as in (**149**).

If the parallel line through the vertex pierces the plane of the base in a point so situated that a tangent to the base cannot be drawn through it, the problem is impossible; if more than one tangent can be drawn, there will be a corresponding number of solutions.

If in this or the preceding problem the line through the vertex be parallel to the plane of the base, the tangent to the base will be parallel to that line. Should that line pierce the plane of the base at a remote and inaccessible point, the tangent plane may be constructed as follows: Through any point of the parallel line pass a plane cutting all the rectilinear elements; and from the assumed point draw a tangent to the curve of intersection. This tangent, and the rectilinear element through the point of contact, will determine the required plane.

152. When the vertex is infinitely remote, a process analogous to that of (**150**) may be employed; in any plane containing the given line and cutting all the rectilinear elements of the cylinder, a tangent to the curve of intersection may be drawn parallel to the given line: the plane of this tangent and the element through the point of contact will be the required tangent plane. This is illustrated in Fig. 146, PC being the tangent and PE the element; and obviously the intersection CE of the tangent plane with the plane of the base will be tangent to the base at E, the foot of the element of contact.

The direct application of this process is not usually convenient; but from it a tentative one is derived, based upon the consideration that if any plane DOF be constructed, containing a line OF parallel to the elements, and another line DO parallel to the given line: then the intersection DF with the plane of the base of the cylinder will be parallel to CE, which line, with the element through the point of contact E, will determine the required plane.

153. The construction in accordance with the preceding argument is shown in Fig. 147, where MN is the given line. Through any point O on any element of the cylinder, as OX, draw a parallel to MN; this line pierces the plane of the base at C, the element pierces it at X, and XC is the line of intersection corresponding to DF of Fig. 146. Parallel to XC draw a tangent to the base; it

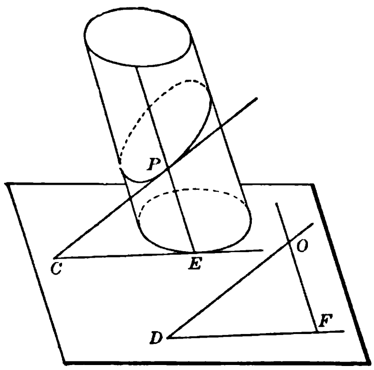


FIG. 146

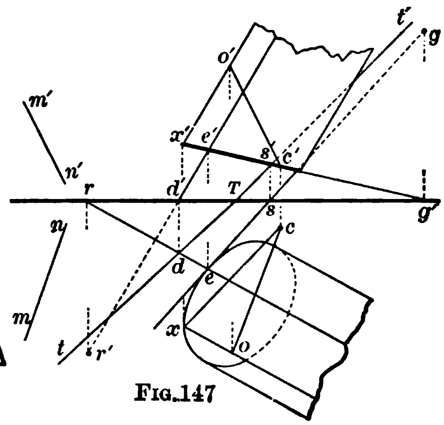


FIG. 147

pierces **V** in S, **H** in G, and E is the point of contact. The element through E pierces **H** in D and **V** in R: joining d and g, then, we have the horizontal trace; joining s' and r', the vertical trace is determined; and these traces meet at T in the ground line.

154. *Special Cases of the Preceding Problems.* In Fig. 148 a cone of revolution is given, with its axis parallel to **AB**: it is required to draw a plane tangent to it, parallel to the given line LM. The parallel through the vertex O pierces **H** in D, **V** in N, and the plane of the cone's base in S. In the profile, the circle of the base is seen in its true form, and S is projected at s_1: the tangent to the base through s_1 pierces **H** at g_1, whose distance from VV determines the distance of g from **AB** in the horizontal projection. Then the points d and g fix the horizontal trace tT, and $Tn't'$ is the vertical trace. If necessary or convenient, the element of contact, seen in the profile as o_1c_1, and in the vertical projection as $o'c'$, may be used to locate points in the traces; for example, the point r' in the vertical trace is found by producing that element.

In the profile, Fig. 149, a cylinder of revolution is shown, its

axis O being parallel to the ground line. This figure sufficiently illustrates the fact that in this case, whether the plane is to be tangent at a given point P on the surface, to pass through a given point R without the surface, or to be parallel to a given line LM, the solution is at once effected and the result clearly exhibited by drawing the profile. The projections on the principal planes are not here given; as in Fig. 74, they would be made up chiefly of a

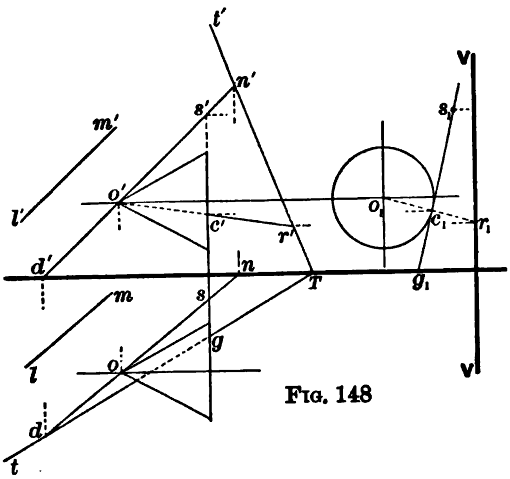


FIG. 148

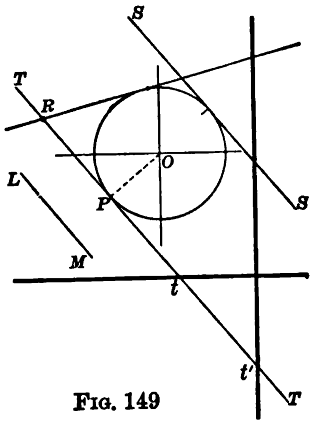


FIG. 149

confusing series of parallels: and all of this is equally true whether the base of the cylinder is circular or of any other form.

155. PLANES TANGENT TO THE HELICAL CONVOLUTE. In Fig. 150 is shown so much of this particular convolute as is necessary to illustrate the application to it of the preceding processes.

In representing it, the helical directrix should be accurately drawn; and this is facilitated by first drawing the front and top views of the cylinder upon whose surface it lies. Now, the axis of this cylinder being vertical and its base lying in **H**, it was shown in (**128**) that the horizontal trace of the convolute will be the involute of the circle of that base; and this should next be carefully constructed, since it is the locus of the points in which **H** is pierced by the rectilinear elements of the surface. Then, om being equal to the quadrant oh, and gn being three times as great, it is clear that OM is tangent to the helix at O, and GN tangent to it at G; and these are the extreme visible elements in the front view of the

portion shown, which embraces three fourths of the circumference of the cylinder, the rectilinear elements terminating, as in Fig. 135, at their points of tangency to the directrix.

156. To assume a point on the surface, assume the horizontal projection, as p. Through p draw the horizontal projection of an element; it is tangent to the circle of the cylinder's base at c, vertically projected at c' on the helix, and cuts the involute at e, vertically projected at e' in **AB**: p' must lie on $c'e'$ the vertical projec-

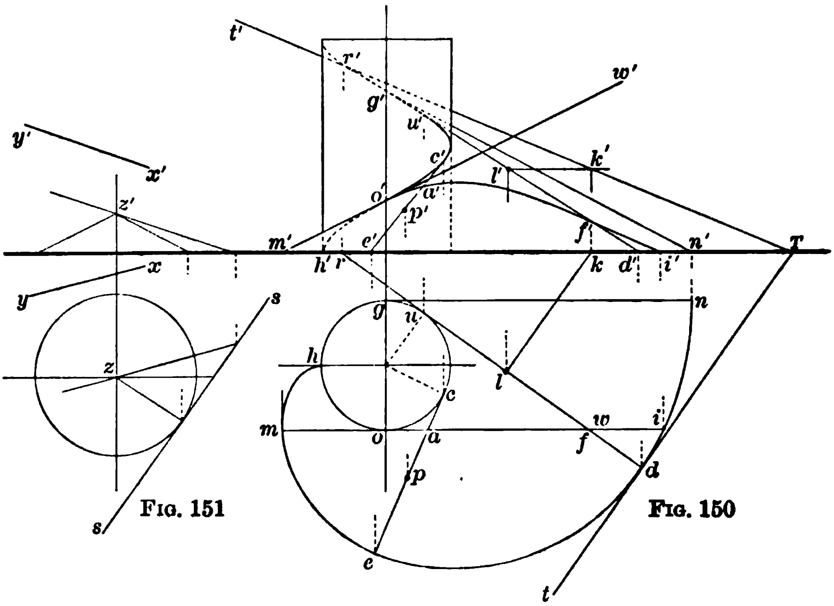


Fig. 151

Fig. 150

tion of the element. By reversing this construction the horizontal projection may be found if p' is assumed.

157. Problem 4. *To draw a plane tangent to this convolute and parallel to a given right line.*

The tangents to the helix make equal angles with the plane of the base, and it is apparent that this surface, though not warped, has a cone directer; which, as shown in Fig. 151, is a cone of revolution, whose elements make the same angle with the plane of the base. If it be required to draw a plane tangent to the convolute and parallel to the given line XY; first draw a plane parallel to this line and tangent to the cone directer (**151**): *ss* is its hori-

zontal trace, and the horizontal trace tT of the required plane is parallel to ss and tangent to the involute. Next draw a tangent to the circular base of the cylinder, perpendicular to tT; this is the horizontal projection of the element of contact, and locates the point of tangency d, which is vertically projected at d' in **AB**. The point of contact u with the circle is projected to u' upon the helix, thus determining $d'u'$ the vertical projection of the element of contact, whose vertical trace is R; and $Tr't'$ is the vertical trace of the tangent plane. Should r' be inaccessible, draw through any point L upon DU a parallel to tT; it is a horizontal line of the plane, and pierces **V** in K, a point of the required vertical trace.

158. *To draw a plane tangent at any assumed point*, as P: the element CE through this point is one line of the plane; the horizontal trace is drawn through e, perpendicular to ce, and the vertical trace is then found as above.

To draw a tangent plane through a point without the surface. Draw first a horizontal plane through the given point (**150**): its intersection with the surface will be another involute, to which a tangent is to be drawn through the given point. This will be one line of the required plane; the element through the point of contact, which is found as in (**157**), is another, and by means of these the traces may be determined.

159. In these illustrations, the lower nappe only of the convolute has been represented and considered, in order to avoid confusion in the diagrams. But it must not be forgotten that the rectilinear elements can be indefinitely extended both ways from the points of contact with the helix; so that a plane tangent to the surface at any point is tangent to it all along a line lying on both nappes: and it makes no difference whether the upper or the lower one is employed in the process of construction.

160. Considering any point as P, Fig. 150, at a given distance CP from the point of contact C between the element and the directrix: it is clear that in the generation of the surface the path of this point will be a helix whose pitch is the same as that of the directrix itself. This particular convolute, therefore, is one of the numerous family of *helicoids*, of which all the others are either warped or double-curved surfaces. And though it is usually pre-

sented and represented in a manner so imperfect and obscure as to conceal the fact, it possesses a certain practical interest, because it is in fact the surface of a screw-thread, which, as will subsequently be shown, can be cut in a lathe in the usual manner.

161. In general, a plane cannot be passed through a given right line and tangent to a single-curved surface. The problem, however, is possible in the following cases, viz., if the given line lies on the convex side of a cylinder, and is parallel to its rectilinear elements; if it passes through the vertex of a cone; or if it is tangent to a line of any single-curved surface.

PLANES TANGENT TO DOUBLE-CURVED SURFACES.

162. A plane tangent to any double-curved surface at a given point can in general be determined (**140**) by drawing at that point a tangent to each of two lines of the surface which pass through that point, the selection depending upon considerations of convenience.

Of the surfaces of this class which are used in the mechanic arts, those of revolution constitute by far the larger portion, by reason of the facility with which they can be formed in the lathe or on the potter's wheel; they serve as well as any for illustrative purposes, and to them we shall confine our attention. In dealing with them in the manner stated, the two curves which would naturally be selected are the meridian line and the circumference of the transverse section through the given point.

163. If a right cone be tangent to a surface of revolution which has the same axis, it will be tangent all round the circumference of a circle. Any plane tangent to this cone will be tangent all along an element; therefore the plane will be tangent to the surface at the point in which this element of contact cuts that circle of contact. Such an auxiliary cone may sometimes be used to advantage in determining a plane tangent to a double-curved surface.

164. If the contour of a surface of revolution be such that the direction of the normal to it can be readily determined, a plane tangent to the surface at a given point can be constructed with great facility, since it is perpendicular to the normal at its extremity.

165. Problem 1. *To draw a plane tangent to a sphere at a given point on the surface.*

Construction. In Fig. 152, let C be the centre of the sphere, and P the given point, assumed as in Fig. 138; then CP is the radius of contact. The horizontal projecting plane of this radius contains a horizontal diameter of the sphere; revolving that plane about this diameter until it is parallel to **H**, P falls at p'', and the

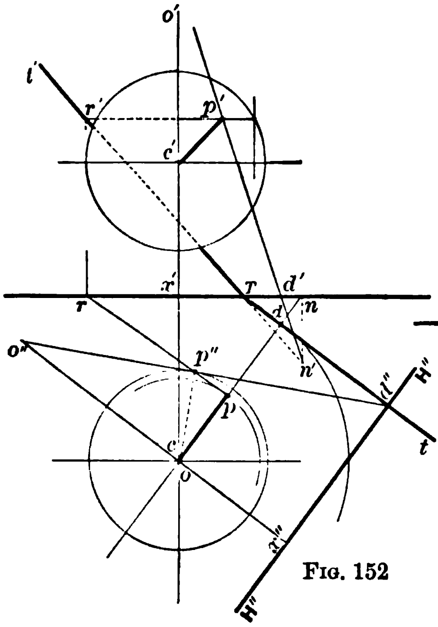


Fig. 152

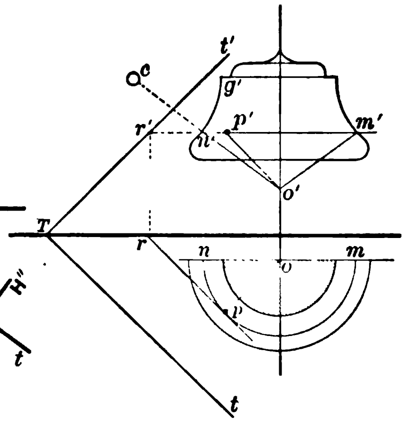


Fig. 153

horizontal trace appears as $H''H''$ parallel to cp, the distance cx'' being equal to $c'x'$. Draw at p'' a tangent to the great circle; it cuts $H''H''$ in d'', the revolved position of a point d in the required horizontal trace, which is perpendicular to cp (**43**). Draw at P a tangent to the horizontal circle of the sphere through that point; it is a line of the tangent plane, and pierces **V** in R: the line PD pierces **V** in N; therefore the points r', n', determine the vertical trace, which must be perpendicular to $c'p'$ (**43**), and meet the horizontal trace at T in **AB**.

Note. Regarding the sphere as a surface of revolution with a vertical axis, it is seen that DPO is an element of a right cone tangent to the sphere around the circle described by P; the base

of this cone in **H** is a circle with radius $cd = x''d''$; and Tt is tangent to this circle.

166. Problem 2. *To draw a plane tangent to any surface of revolution at a given point on the surface.*

Construction. In Fig. 153, the assumed point P lies upon the circle whose vertical projection is $m'n'$. Let c be the centre of curvature of the contour $n'q'$, then cn' is normal to the curve at n'; its prolongation cuts the axis in O, and $o'n'm'$ is the vertical projection of a cone normal to the given surface. Therefore $o'p'$ is the vertical and op is the horizontal projection of the normal to the surface at P. Through P draw, as in Fig. 67, a plane tTt' perpendicular to PO; it is the tangent plane required.

167. Problem 3. *To draw through a given line a plane tangent to a given sphere.*

First Method. **Analysis.** Regarding the given line as the intersection of two planes tangent to the sphere, each plane is perpendicular to a radius at its extremity; therefore the plane of these two radii is perpendicular to the given line. If then a plane be passed through the centre of the sphere and perpendicular to the given line, and from the point in which it cuts the line a tangent be drawn to the great circle cut from the sphere; then the plane determined by that tangent and the given line is the one required.

Construction. Let C, Fig. 154, be the centre of the given sphere, MN the given line.

On the horizontal projecting plane of the line make a supplementary projection; in this $H'H'$ is the horizontal plane, c_1 the centre of the sphere, m_1n_1 the given line, and LL the plane perpendicular to the line, of which ll is the horizontal trace. This plane contains a horizontal diameter of the sphere, of which ge is the horizontal projection; and it cuts MN in the point A. Revolve the plane about ge until it becomes horizontal; its trace ll then appears as $l''l''$, and a falls at a''. Draw $a''d''$ tangent to the great circle of the sphere, and find o'' the point of contact. Making the counter-revolution, a'' returns to a, d'' falls at d in ll (vertically projected at d' in **AB**), and o'' goes to o on da, whence it is vertically projected to o' on $d'a'$.

The tangent line DA pierces **H** in D, the given line pierces it

in N; therefore the points d and n determine the horizontal trace tT, which must also be perpendicular to co: MN pierces V in R, and r' is a point in the vertical trace Tt' of the required tangent plane; and this trace must also be perpendicular to $c'o'$, the vertical projection of the radius of contact.

168. *Second Method.* **Analysis.** If any point of the line be taken as the vertex of a cone tangent to the sphere, a plane con-

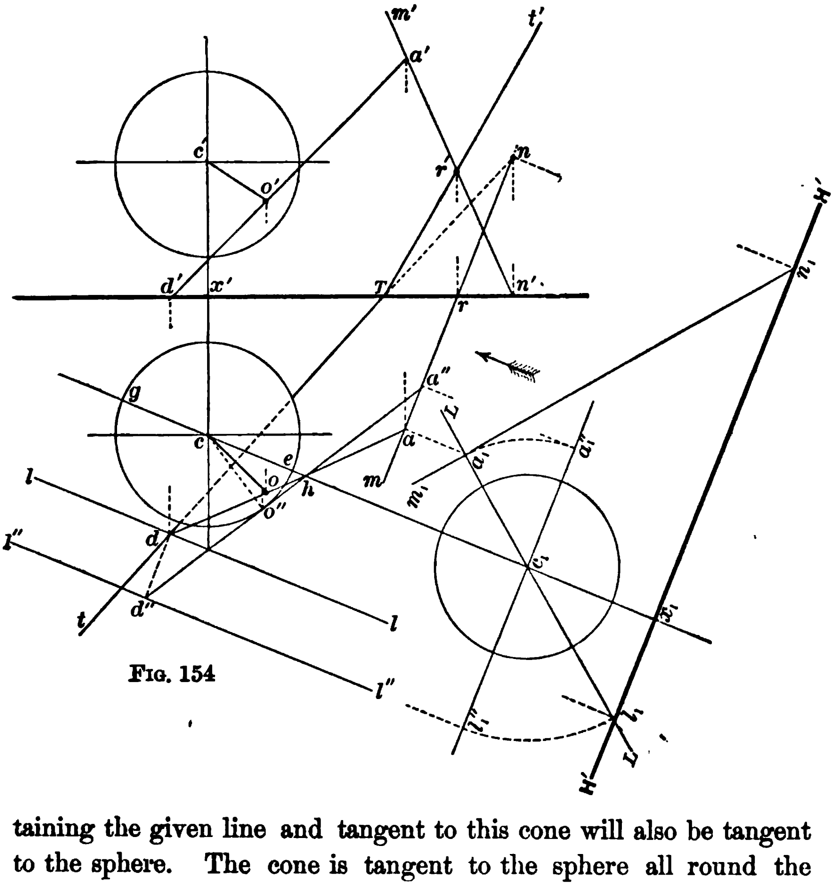


Fig. 154

taining the given line and tangent to this cone will also be tangent to the sphere. The cone is tangent to the sphere all round the circumference of a small circle; the plane is tangent to the cone all along an element; and the intersection of the circle and the element is the point of contact between the sphere and the plane.

Construction. In Fig. 155, C is the centre of the sphere, MN the given line. The point D, in which the line is cut by a hori-

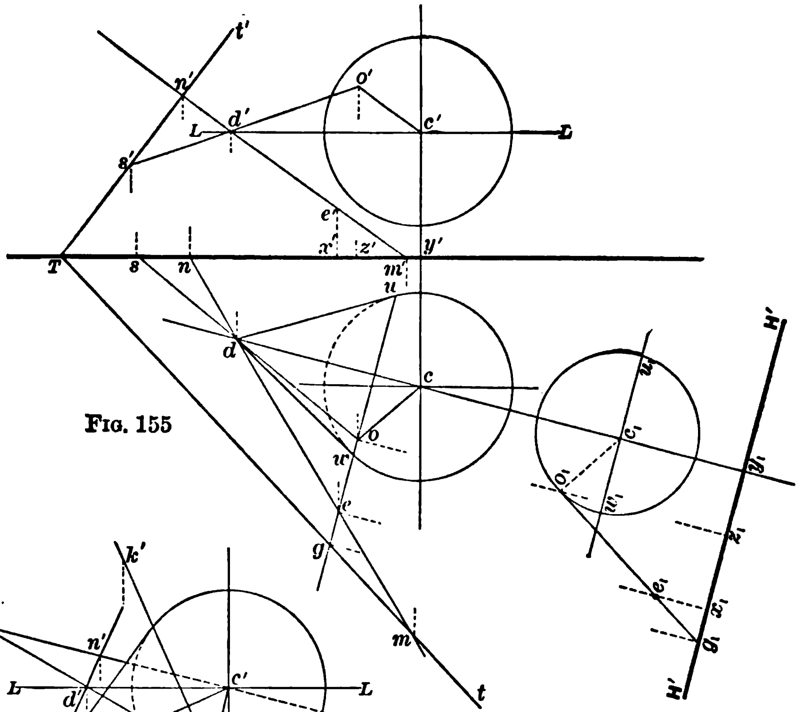


FIG. 155

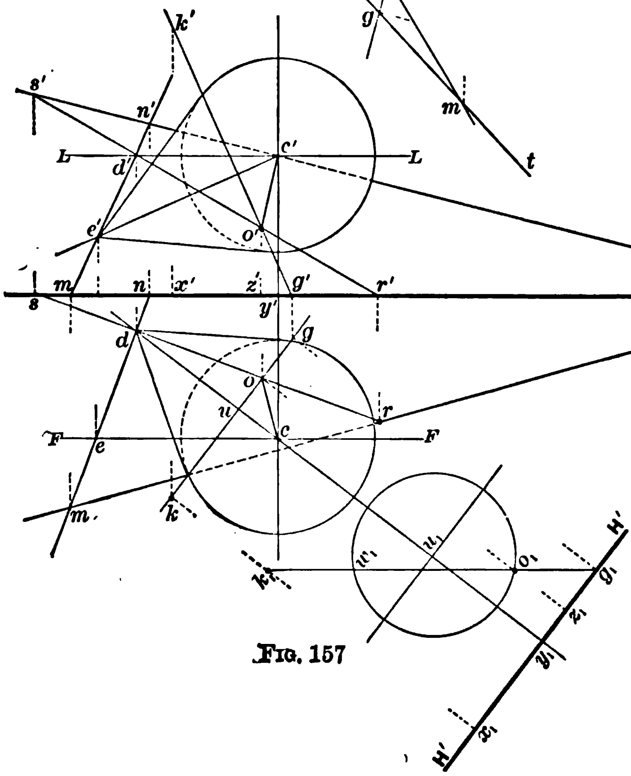


FIG. 157

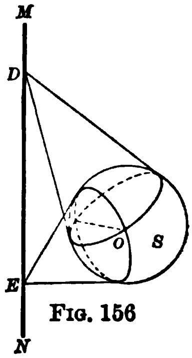


FIG. 156

zontal plane LL through C, is selected as the vertex of the auxiliary cone, merely for convenience; its axis being thus made horizontal, the plane of the circle of contact is vertical, and appears as a right line uw in the horizontal projection. This plane cuts MN in a point whose horizontal projection is e and vertical projection e'; in the supplementary projection on this plane it appears as e_1: through e_1 draw a tangent to the circle of contact, find the point of tangency o_1, and produce it to cut $H'H'$ in g_1. These two points are projected back to o and g; g is one point, and m is another, in the horizontal trace of the required plane, which must also be perpendicular to co. The vertical projection o' will be in a perpendicular to **AB** through o, at a distance $o'z'$ above **AB** equal to o_1z_1 in the supplementary projection; the given line pierces **V** in N, and the element OD of the cone pierces it in S; therefore $Ts'n't'$ is the vertical trace of the tangent plane, which must be perpendicular to $c'o'$, the vertical projection of the radius of contact. In order to avoid confusion, the vertical projection of the auxiliary cone is not drawn, no use being made of it in the construction.

169. *Third Method.* **Analysis.** If any two points of the given line be taken as the vertices of cones tangent to the sphere, the planes of the circles of contact will intersect in a common chord, whose extremities will lie on all three surfaces. A plane containing the given line and either of these points will be tangent to the sphere.

Construction. This is illustrated pictorially in Fig. 156, where S is the sphere, MN the given line, and D, E, are the vertices of the two cones. The construction in projection is given in Fig. 157, C being the centre of the sphere, and MN the given line. A horizontal plane LL through C cuts the line in D, and a plane FF through C and parallel to **V** cuts it in E. Take D and E as the vertices of the two cones; then the axis of the first being horizontal, the plane of the circle of contact will be vertical; kg is its horizontal trace. The axis of the second will be parallel, and the plane of the circle of contact perpendicular, to the vertical plane; $k'g'$ is the vertical trace. Therefore kg, $k'g'$, are the projections of the intersection of these two planes. This line pierces **H** in G; in order to find where it pierces the surface of the sphere, make a

supplementary projection on a plane perpendicular to the axis of the first cone. In this projection the line of contact is seen in its true form as the circle of which u_1 is the centre, and the line k_1g_1 cuts its circumference at o_1, which is projected back to o, and thence vertically to o', the distance $o'z'$ being equal to o_1z_1. Then DO is a line of the required plane; its vertical and horizontal traces are S and R; those of the given line are N and M; therefore mrT is the horizontal and $Tn's'$ is the vertical trace of the tangent plane: and these traces are respectively perpendicular to the projections of CO the radius of contact.

The vertical projection of the first cone and the horizontal projection of the second are omitted; no use being made of either in effecting the solution, their introduction would simply confuse the diagram. A like result would have followed had the determination of the other tangent plane been represented, and the construction of one is sufficient to illustrate the principles of either method.

170. PROBLEM 4. *To find the point of contact of any surface of revolution with a plane perpendicular to a given line.*

Construction. In Fig. 158, the axis of the surface is vertical.

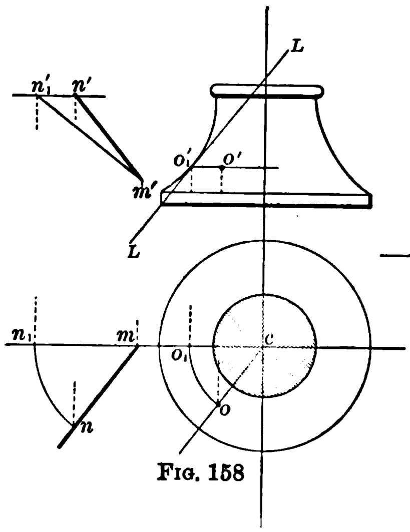


FIG. 158

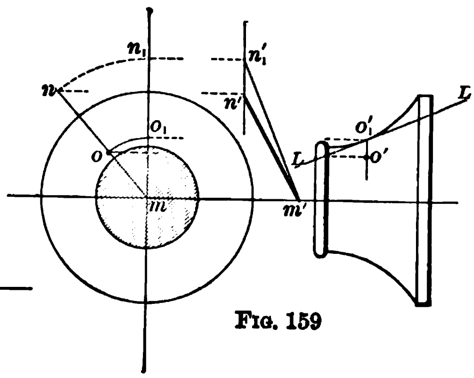


FIG. 159

No ground line is drawn, but the axis may be regarded as lying in a vertical plane, represented in the top view by the horizontal cen-

tre line. Revolving the given line MN about a vertical axis until it lies in this plane, it takes the position MN_1. The plane LL is drawn tangent to the contour and perpendicular to $m'n_1'$, and O_1 is the revolved position of the point of contact. In the counter-revolution this point describes an arc of a horizontal circle, the angle o_1co being equal to the angle n_1mn, and O is the true position of the required point.

In Fig. 159, the axis is horizontal, and a vertical plane containing it is represented by the vertical centre line in the end view. The given line MN and the surface together are revolved about the axis until the line comes into that vertical plane; and the remaining steps require no explanation.

In Fig. 160, the axis lies in the vertical plane, but is inclined, the object accordingly appearing foreshortened in the top view. The line to which the tangent plane is to be perpendicular is given

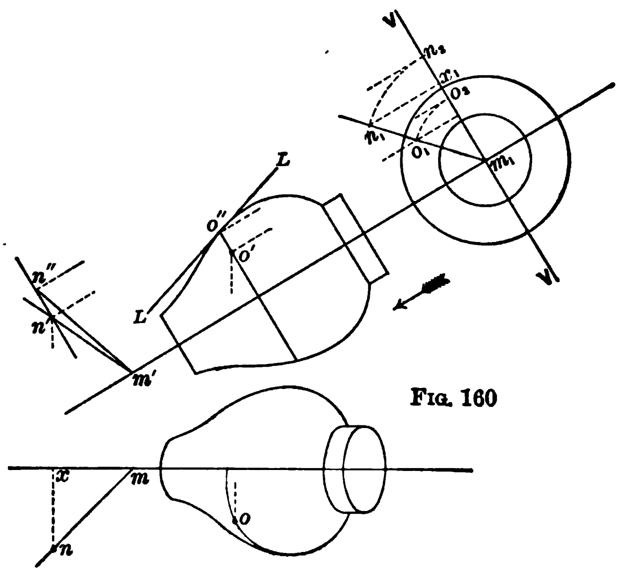


FIG. 160

by the projections mn, $m'n'$; in the supplementary end view it appears as m_1n_1, the distance n_1x_1 being equal to nx in the top view. This line is revolved about the axis of the surface into the plane VV as in Fig. 159, n_1 going to n_2, which is projected back to n'';

the plane LL is then drawn tangent to the contour, and the point of contact o'' is projected to o_2, revolved to o_1, and re-projected to o'. In the top view, o is directly under o', at a distance from the centre line equal to the distance of o_1 from VV in the end view.

NOTE. The determination of the outline in the top view, Fig. 160, depends upon this principle, viz., that **the visible contour of any object** *is the envelope of all the lines upon its surface.* In the present case, all the transverse sections are circles, whose horizontal projections are ellipses, and the contour is **tangent** to all these ellipses.

171. PROBLEM 5. *To draw a plane making given angles with the principal planes of projection.*

Argument. In the profile, Fig. 161, O and P are the vertices of two cones tangent to the same sphere, the axis of one being ver-

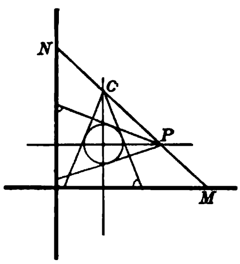


FIG. 161

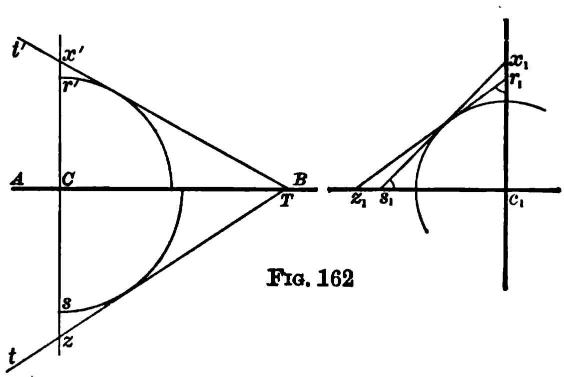


FIG. 162

tical, that of the other perpendicular, to **V**. All planes tangent to the first are equally inclined to **H**; all those tangent to the second are equally inclined to **V**: a plane through the line MN joining their vertices, if tangent to one, will be tangent to both, as in Fig. 157. M will be a point in the horizontal trace, N a point in the vertical trace; and each trace will be tangent to the base of the cone which lies in the corresponding plane of projection.

In the application, the angles at the bases of the cones must be made equal to the assigned angles which the required plane is to make with **H** and **V** respectively.

Construction. In Fig. 162, the profile is first drawn, the cen-

tre c_1 of the sphere being for convenience placed in the ground line. Tangent to the outline of the sphere, draw x_1s_1, making the angle $x_1s_1c_1$ equal to the assigned angle with **H**; and also z_1r_1, the angle $z_1r_1c_1$ being equal to the assigned angle with **V**. Then draw an indefinite perpendicular to **AB** through any convenient point C, and on it set off Cx' above, equal to c_1x_1; also Cz below, equal to c_1z_1. About C describe above **AB** an arc with radius $Cr' = c_1r_1$, and draw $x'T$ tangent to it; also about C describe below **AB** an arc with radius $Cs = c_1s_1$, and draw zT tangent to it. These two tangents will be the traces of the required plane, which must intersect at T in the ground line.

NOTE. A different solution of this problem has already been given in (**94**), but the one here presented as a neat application of tangent planes is preferable, being more reliable as well as more expeditious.

CHAPTER V.

OF INTERSECTIONS AND DEVELOPMENTS.

Intersection of Surfaces by Planes. Development of Single-curved Surfaces. Tangents to Curves of Intersection, before and after Development. Problem of the Shortest Path. Intersections of Single-curved Surfaces. Infinite Branches. Intersections of Double-curved Surfaces. Intersection of a Cone with a Sphere. Development of the Oblique Cone.

172. If a line be drawn upon one surface, the point in which it pierces any other surface will lie upon the intersection of the two. In order to find that point, an **auxiliary surface** is passed through the line; this intersects the second surface in another line, which in turn cuts the first line in the point sought.

This is illustrated in finding the point in which a given right line pierces a plane; the **auxiliary plane** intersects the given plane in another right line, and that cuts the first one in the required point. It is clear that the line here spoken of as *given* might have been cut from **any given ruled surface** by the auxiliary plane; then the point, located as above, would have been one point in the **intersection of that surface by the given plane**; and other points could be found by means of other auxiliary planes.

Again, were any other surface substituted for the given plane, the construction would be modified only in this, that the auxiliary plane might intersect it in a curve instead of in a right line: but this curve would still cut the rectilinear element of the first surface in a point of the line of intersection of the two surfaces.

173. The point last mentioned is the one in which the first surface is pierced by the curve cut from the second; and evidently would be, whether the line cut from the first by the auxiliary plane were straight or otherwise. Which shows that the point in which any plane curve pierces a given surface may be found by first de-

termining the intersection of its plane with that surface; it will be a line cutting the given curve in the required point.

But even in the case of a right line piercing a plane it is not necessary that the auxiliary surface should be a plane: if, for example, a cone or a cylinder be passed through the line, intersecting the given plane in a circle, it will now be apparent that the circumference will cut the given line in the point required. And in dealing with a line of double curvature the simplest auxiliary surface that can be used is cylindrical. Thus, in order to find where the curve DEG, Fig. 131, pierces the vertical plane, we make use of the horizontal projecting cylinder, from which **V** cuts the element kk', and this intersects the curve in k', the point sought.

The same figure illustrates the process of finding the intersection of two cylindrical surfaces; one, whose base is cek, having vertical elements, while the elements of the other, whose base is $c'e'k'$, are perpendicular to **V**. An auxiliary plane ee', parallel to the rectilinear elements of both cylinders, cuts from each a right line, and these intersect in E; and by other planes parallel to this the points D and G are determined; these points lie upon both surfaces, and the curve cEk' is the required intersection.

174. From the preceding it will be perceived that the problems of finding the points in which surfaces are pierced by lines, and the lines in which surfaces intersect each other, are correlated and interwoven, involve the same principles, and require for their solution a previous knowledge of the intersections of the surfaces under consideration by certain others, which may be used as auxiliaries. In all of them the ultimate determinations consist in the location of points by the intersections of lines; and the auxiliary surfaces should be so selected that these lines shall cut each other as nearly as possible at right angles.

175. Tangents to Curves of Intersection. If a surface is cut by a plane, the tangent to the line of intersection at any point will lie in that plane, and also in the plane tangent to the surface at the given point; therefore it will be the intersection of those planes.

If two surfaces intersect, the tangent to the common line at any point will be the intersection of two planes, one tangent to each surface at that point.

NOTE. If, as sometimes happens, the two planes whose intersection should determine the tangent coincide, this method obviously fails to give any result. In this event, the direction of the tangent can be determined only by methods depending upon the mathematical properties of the curve, of which this branch of science does not treat.

176. Development of Surfaces. A prism is capable of rolling, in a hobbling and imperfect manner, upon a plane; turning about each edge in succession as an axis, so that one face after another is brought into coincidence with the plane. If the number of edges be increased, the hobbling will diminish, and when the number becomes infinite, it will disappear entirely: the rolling is perfect, the change from one axis to another going on continuously, since the edges are now consecutive elements of a cylinder. In this way all the elements of the cylinder, without change of relative position, can be brought into the plane, the area rolled over being equal to that of the surface which has rolled over it.

A pyramid, treated in like manner, ultimately becomes a cone, which possesses the same property: indeed it is a sufficiently familiar fact that either of these surfaces can be unrolled into a plane, without extension, compression, or distortion of any kind.

177. This process is called the **development** of the surface. And the two illustrations above given show clearly upon what its possibility depends: the plane of development is not only tangent to the surface, but contains two of its consecutive elements, and therefore the elementary surface included between them. These elements must be rectilinear, since each in turn is an axis of rotation, and an axis is a right line; no two consecutive elements of a warped surface lie in the same plane, therefore **all single-curved surfaces**, and no others, are capable of development.

It is evident that if any surface can roll upon a plane, the plane is equally capable of rolling upon the surface; and this development by rolling on a fixed plane is, in a limited sense, the converse of the generation of the surface by a moving plane.

But in order to execute this process, it is not enough to know that the surface is developable: it is necessary also to know beforehand the developed forms of one or more lines of the surface which

intersect the rectilinear elements, in order to determine the relative positions of these elements on the plane of development.

INTERSECTIONS OF SURFACES BY PLANES.

178. Problem 1. *To find the intersection of a right circular cylinder with a plane.*

Construction. In Fig. 163, the axis of the cylinder is vertical, its base lies in the horizontal plane, and the given plane tTt' is perpendicular to V.

The points in which the elements through the points x, 1, 2, etc., of the base pierce the plane are seen directly at u', $1'$, $2'$,

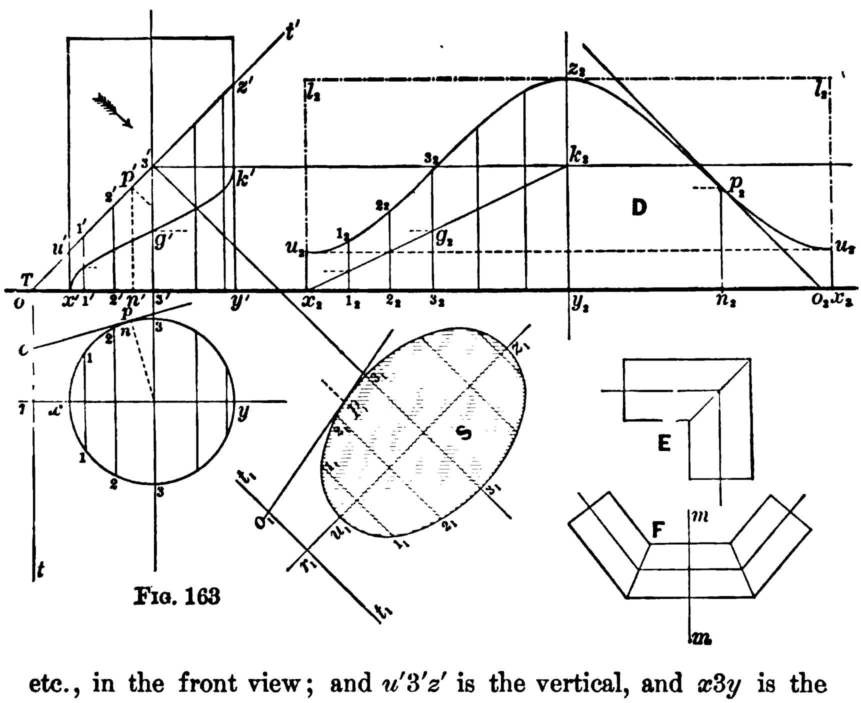


Fig. 163

etc., in the front view; and $u'3'z'$ is the vertical, and $x3y$ is the horizontal, projection of the required intersection.

To draw the tangent at any point P. This must lie in the tangent plane, which is vertical, and has the horizontal trace po tangent to the circumference of the base; the tangent also lies in the cutting plane, therefore its vertical projection $p'o'$ coincides with Tt'.

To show the true form of the curve. Make the supplementary projection S, looking perpendicularly against the given plane. In this view, tt will appear as t_1t_1 perpendicular to Tt', $u'z'$ will be seen in its true length as u_1z_1' parallel to Tt', and the chords vertically projected at 1′, 2′, etc., will also be seen in their true lengths as 1_11_1, 2_12_1, etc.; these lengths are equal to those of 11, 22, etc., in the top view; this curve is an ellipse whose major axis is $u'z'$, and whose minor axis is equal to 33, the diameter of the given cylinder.

To draw the tangent to this curve in its own plane. The point whose horizontal projection is p and vertical projection p' is found in the supplementary projection at p_1: the tangent at this point was seen in the top view to cut the horizontal trace at o, and xy produced cuts that trace in r. In the supplementary projection z_1u_1 produced cuts the horizontal trace in r_1; now set off $r_1o_1 = ro$, and draw p_1o_1; it will be the required tangent at the point p_1.

179. To develop the lower part of the cylinder. Suppose the cylinder, formed of thin sheet metal, to be cut through the element $x'u'$, and unrolled. The vertical elements will remain vertical, and since the base is a continuous curve perpendicular to them all, it will develop into a right line x_2x_2, as shown in D, whose length is equal to the circumference. If in the top view this circumference is divided into equal parts at 1, 2, etc., then the developed base will be similarly divided at 1_2, 2_2, etc.; at each point of subdivision there will be a vertical ordinate, representing an element; and since the lengths of these elements remain unchanged, we shall have $1_21_2 = 1'1'$, $2_22_2 = 2'2'$, etc., and $u_23_2z_2u_2$ will be the development of the curve of intersection.

To draw the tangent to the developed curve. The tangent to the intersection at P contains two consecutive points of the curve, and will contain them after development, and will therefore be tangent to the developed curve. When the plane of development becomes tangent to the cylinder at P, it will contain the element PN, the tangent PO, and the subtangent NO; and these will remain unchanged in magnitude and in relative position. In the development D this element falls at p_2n_2; therefore, setting off $n_2o_2 = no$, we have p_2o_2 as the tangent to the developed curve.

In drawing the tangents at u_2 and z_2 in this manner, it is

obvious that the subtangents will be infinite; the tangents at those points are therefore parallel to x_2x_2.

180. The Problem of the Shortest Path. Let it be required to find the shortest path on the surface of the cylinder, between the points x, x', and y, k'. In the development these points fall respectively at x_2, k_2, and the least distance between them is the right line which joins them. This line cuts the various elements at points whose distance from the base will remain the same when the developed sheet is re-formed into a cylinder. These points, therefore, are projected back to the original positions of the elements, as k_2 to k', g_2 to g', x_2 to x', etc., thus forming in the vertical projection the curve $x'g'k'$, which represents the required shortest path. In the development **D** it is seen that the distances of the points of this curve from the base of the cylinder, measured on equidistant elements, increase at a uniform rate; therefore the curve itself is a helix, of which $y'k'$ is half the pitch. In the vertical projection the outer elements of the cylinder are tangent to the curve x' and k'; g' is a point of contrary flexure, and the tangent at that point is parallel to x_2k_2.

It is also to be noted that the development of the curve of intersection is identical with the projection of a helix whose half-pitch is equal to x_2y_2, lying on a cylinder of which the diameter is u_2l_2; the tangent at 3_2 is parallel to $u'z'$.

181. Practical Applications. Since the ellipse is perfectly symmetrical about both axes, it may be turned end for end; thus the two portions of a cylinder cut by a plane making an angle of 45° with the axis may be joined together as shown at **E**, making what is known as a "square elbow." By using other angles, the pieces may be put together at different inclinations, as shown at **F**, which represents a "three-section elbow." In order to lay out the sheet for the middle piece, cut it by a plane mm perpendicular to the axis; this section will develop into a right line as in **D**, and the ordinates are set off each way from this to determine the contour.

These problems of development are of direct use to workers in sheet metal. In theory it makes no difference along what element the surface is cut; but in practice, the rule dictated by plain com-

mon sense is to cut it *so as to make the shortest seam*, unless there is some good reason for doing otherwise.

182. PROBLEM 2. *To find the intersection of an oblique cylinder by an oblique plane.*

Construction. In Fig. 164, the plane and the cylinder are each

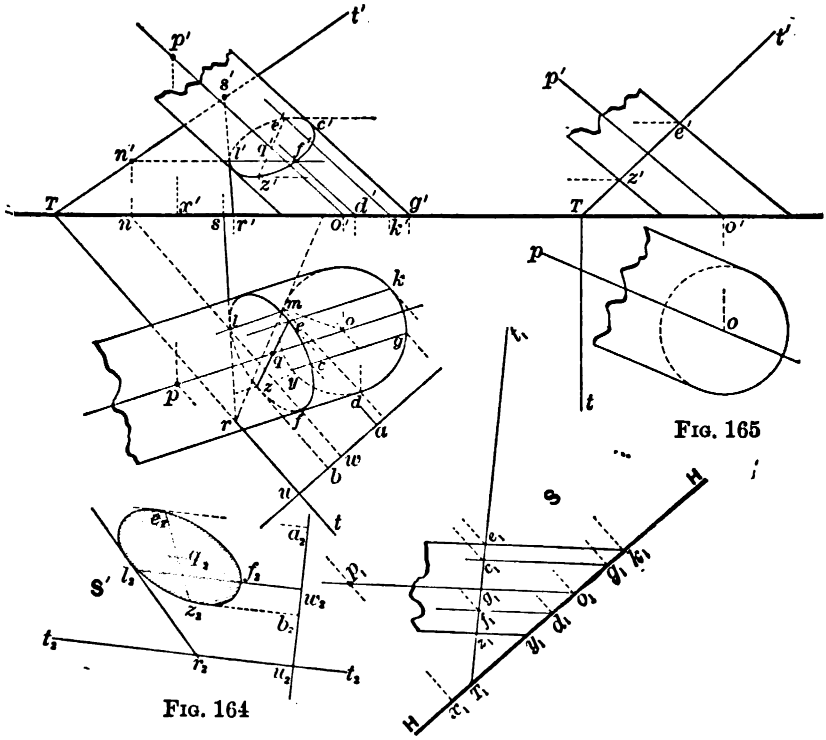


FIG. 165

FIG. 164

inclined to both **H** and **V**. Make a supplementary projection **S**, looking in the direction tT; the horizontal plane appears as $H'H'$ perpendicular to tT, and the given plane as T_1t_1, both planes being seen edgewise and at their true inclination to each other. The centre O of the base is here projected at o_1, and any point P of the axis at p_1, the altitude p_1x_1 being equal to $p'x'$; then the new projections of the elements are parallel to o_1p_1. The extreme visible elements are determined by drawing at k and y in the horizontal projection, tangents to the base, perpendicular to $H'H'$; drawing, through k_1 and y_1, parallels to o_1p_1, these lines are cut by T_1t_1 at

e_1 and z_1, which obviously are the highest and lowest points of the curve: e_1 is projected back to e on the horizontal projection of the element through K, and thence upward to the vertical projection of the same element, e' being as far from **AB** as e_1 is from $H'H'$. The positions of z and z' are determined in the same manner, and by repeating the process a point may be found on any element at pleasure: it is particularly desirable to locate with accuracy those points which lie upon the limiting elements, as f in the horizontal and c' in the vertical projection, since they are points of tangency.

183. The above operation might be defined as consisting in the use of a system of auxiliary planes, parallel to the cylinder and to the horizontal trace of the given plane; these cut horizontal lines from the plane and elements from the cylinder, whose intersections are points in the required curve. Thus in the supplementary projection **S**, d_1f_1 may be regarded as representing a plane perpendicular to the paper; its horizontal trace is dm, and its intersection with tTt' is a line horizontally projected as fl, piercing **V** in N, and vertically projected in $n'f'$ parallel to **AB**. The horizontal trace cuts the base in d and m, and the horizontal projections of the elements through these points determine f and l, vertically projected at f' and l'; and these are points in the required curve.

184. To draw a tangent to the curve of intersection. Let L be the point at which the tangent is to be drawn. The plane tangent to the cylinder at this point will contain the element whose horizontal projection is lm; and its horizontal trace, tangent to the base at m, cuts Tt in r, vertically projected at r' in **AB**: therefore RL is the required tangent, which if produced must pierce **V** in a point of the vertical trace of the given plane. This point may be determined by producing rl to cut **AB** in s, which is its horizontal projection, and s' in Tt' is its vertical projection.

To show the curve and its tangent in their own plane. Make a second supplementary projection **S'**, looking perpendicularly towards T_1t_1; the horizontal trace will appear as t_2t_2 perpendicular to T_1t_1, and the different points of the curve will lie upon projecting lines drawn through e_1, z_1, etc., parallel to t_2t_2. In order to determine their relative positions, draw in the horizontal projection any line

of the plane perpendicular to Tt, as au; this will be seen in **S'** as $a_s u_s$ perpendicular to $t_s t_s$, and the distances of the points of the curve from this line will be the same as their distances from au in the horizontal projection: thus, $a_s e_s = ae$, $b_s z_s = bz$, $w_s f_s = wf$, etc.

To draw the tangent at l_s: set off on $t_s t_s$ the distance $u_s r_s = ur$, and draw $r_s l_s$; it is the tangent required.

185. From (**179**) it is apparent that the vertical and horizontal projections might have been constructed by means of the auxiliary planes, without using the supplementary projection **S**; and indeed any other system of planes might have been substituted for the one here employed. In either case, a rigid test of the accuracy of the construction would be to revolve the plane and cylinder together around a vertical axis until, as in Fig. 165, the horizontal trace becomes perpendicular to **AB**; the vertical projections of all points in the curve should then fall in one straight line, Tt'. By the use of the projection **S** this test is applied at the outset; the construction of the projections on **H** and **V** is facilitated; and the true form of the curve is found by making the projection **S'**, far more readily than it can be in any other way.

186. In order to develop the cylinder, cut it by a plane perpendicular to the rectilinear elements; this right section will develop into a right line. On this line set off the rectified distances between the elements, and perpendicular to it draw the elements themselves; on these perpendiculars lay off **the true distances,** as measured on the elements, from the right section to any line of the surface whose developed form may be required.

To draw a tangent to such a developed curve. Draw first a tangent to the curve in its original position, measure the angle between it and the element passing through the point of contact, and draw a line making the same angle with that element at the corresponding point on the developed sheet.

187. PROBLEM 3. *To find the intersection of an oblique cone by an oblique plane.*

Construction. This differs from that above explained, merely in the respect that the elements are convergent instead of parallel. The point in which any element pierces the plane, is seen directly

in the vertical projection, Fig. 166, since the plane itself is perpendicular to **V**; and the horizontal projection must of course lie on the horizontal projection of the element. The true form of the

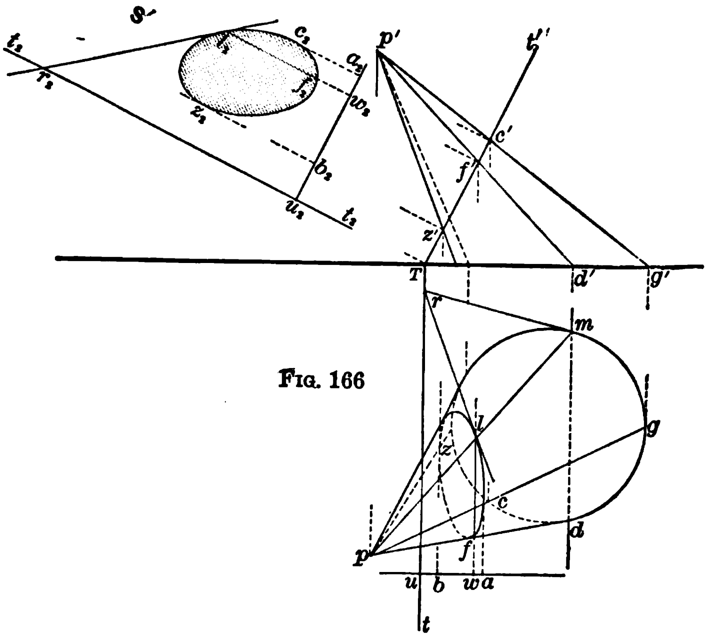


FIG. 166

curve is found by constructing the supplementary projection **S'**, precisely as in Fig. 164; and the diagrams being lettered as nearly as may be to correspond, no detailed explanation is necessary.

The determination of the tangent at L, in the horizontal as well as in the supplementary projection, is also made in the same manner as in the case of the cylinder.

The development of such a cone cannot now be explained; because no method has as yet been described of drawing a line upon the surface which will develop into a curve of known form: this matter will be discussed subsequently.

188. PROBLEM 4. *To find the intersection of a cone of revolution by a plane.*

Construction. In Fig. 167, the axis of the cone being vertical and tTt' perpendicular to **V**, the vertical projections u', e', z', etc.,

of the points in which the plane cuts the rectilinear elements are seen by inspection of the front view, and may be thence projected to u, e, z, etc., in the top view. But in this top view the projecting lines would cut the elements very acutely in the neighborhood of FP, thus making the determinations unreliable. In that region another process is preferable: the element FP for instance pierces tTt' at a point whose vertical projection is n'; a horizontal plane through that point will cut the cone in a circle whose radius is in', and its intersection with the given plane will be a right line per-

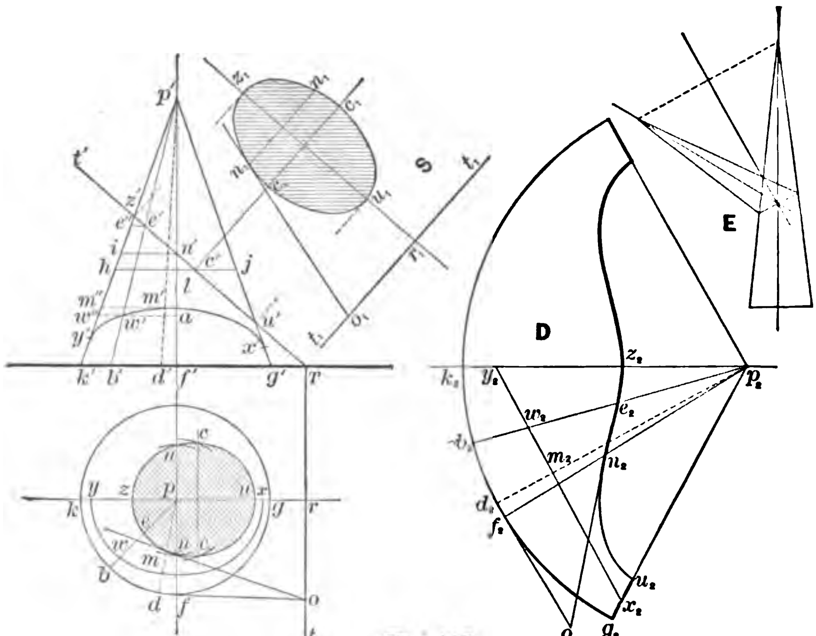


FIG. 167

pendicular to **V**, seen in its true length in the top view as a chord nn in the circle cut from the cone; and this process may be repeated to determine other points.

189. Since the given plane in this case cuts all the elements, the intersection is an ellipse, the true length of whose major axis is $u'z'$; the horizontal projection is also an ellipse, whose major axis is uz. In order to determine the minor axis, bisect $u'z'$ at c', through which point draw a horizontal plane hj; this cuts the cone

in a circle whose radius is lh, and c' is horizontally projected as a chord cc of the circle described about p with this radius. It is perfectly legitimate, and practically it is preferable, thus to determine the axes, and to construct the curve by any of the well-known methods of drawing the ellipse, not only in the horizontal projection, but in the supplementary projection **S**, where the intersection is seen in its true size.

The same modes of finding points in the curve may be used when the plane cuts a parabola or an hyperbola from the cone; the use of auxiliary transverse sections being more particularly applicable when the angle at the vertex is acute.

190. To draw a tangent at any point of the intersection, as N. The element through this point pierces **H** at F, and fo, tangent to the circle of the base, is the horizontal trace of the tangent plane; this cuts Tt in o, and on is the horizontal projection of the tangent. A plane through the axis, parallel to **V**, cuts Tt in r; in the supplementary projection **S**, set off $r_1o_1 = ro$, and o_1n_1 will be the tangent to the curve in its own plane, as in Fig. 163.

If the given plane be parallel to the plane of any two elements, the section will be an hyperbola; the plane tangent to the cone along either of these two elements, will intersect the cutting plane in a line parallel to the element itself, and this line will be an asymptote to the curve. **If the cutting plane be parallel to one element only,** it will be parallel to the plane tangent along that element, and therefore will not intersect it at all; which is as it should be, since the curve of intersection is a parabola, which has no asymptote.

191. To develop this cone. Since every point in the base is equidistant from the vertex, the base itself will develop into an arc of a circle whose radius is the slant height; and the length of this arc will be equal to the circumference of the base. Thus in the diagram **D**, the arc g_2k_2, described with radius $p_2g_2 = p'g'$, is equal in length to the semi-circumference qfk. This semi-circumference is bisected at f; and bisecting g_2k_2 at f_2, we have f_2p_2 as the position of the element FP on the developed sheet: and in like manner the position of any other element may be found.

To find the developed form of the intersection: Lay off on the

elements in the development, the **true distances** from the vertex to the points in which these element pierce the plane tTt'. Thus, $p_1u_1 = p'u'$, and $p_1z_1 = p'z'$: the element PF pierces the plane at N, but $p'n'$ is a foreshortened view of a line whose true length is $p'i$, found by passing a horizontal plane through N; therefore on p_1f_1 make $p_1n_1 = p'i$, and n_1 will lie on the required curve; also $p_1e_1 = p'e''$, and so on.

To draw a tangent to the developed curve, at any point as n_1. In the horizontal projection the tangent at n is no, the hypothenuse of a right-angled triangle fno, which triangle lies in the tangent plane, and will in the development be seen in its true size and form. Therefore, draw f_1o_1 perpendicular to f_1p_1 and equal to fo; then o_1n_1 is the required tangent. The principle is, that the tangent makes the same angle with the element after development as before; hence the tangents at u_1 and z_1 are perpendicular respectively to g_1p_1 and k_1p_1.

192. To find the shortest path on the surface, between any two points as X and Y. These points fall at x_1 and y_1 in the development; the right line x_1y_1 is the developed path; it cuts p_1b_1 at w_1; set off $p'w'' = p_1w_1$, draw through w'' a horizontal line cutting $p'b'$ in w', the vertical projection of a point in the required curve: the horizontal projection is best found by setting off pw equal to aw''; and in like manner, other points may be determined. The highest point M is found by drawing p_1d_1 perpendicular to x_1y_1, which it cuts at m_1: make the arc fd equal to the arc f_1d_1, and project d to d'; then the position of M on PD is ascertained as above.

Note. The problem of the shortest path, on any surfaces which can be developed either by unrolling, as in the case of single-curved surfaces, or by unfolding, like prisms or pyramids, is always solved in this manner, provided that a right line *can* be drawn on the developed sheet from one point to the other, which, however, is not always possible.

193. It is clear that, as in the case of the cylinder, the ellipse may be turned end for end, and the upper part of the cone joined to the lower as shown in the small diagram **E**; in going from the old position to the new, the vertex describes a semicircle about an axis perpendicular to the elliptical section at its centre.

194. Problem 5. *To develop the helical convolute.*

Analysis. If through the tangent to the directrix at any point a plane be passed containing the centre of curvature at that point, the helix can be rolled upon that plane portion by portion; and, the radius of curvature being constant, it will thus develop into a circle having that radius. The rectilinear elements of the surface being tangents to the helix, will appear in the development as tangents to this circle.

Construction. Let it be required to develop so much of the lower nappe of the convolute shown in Fig. 168, as lies above **H**,

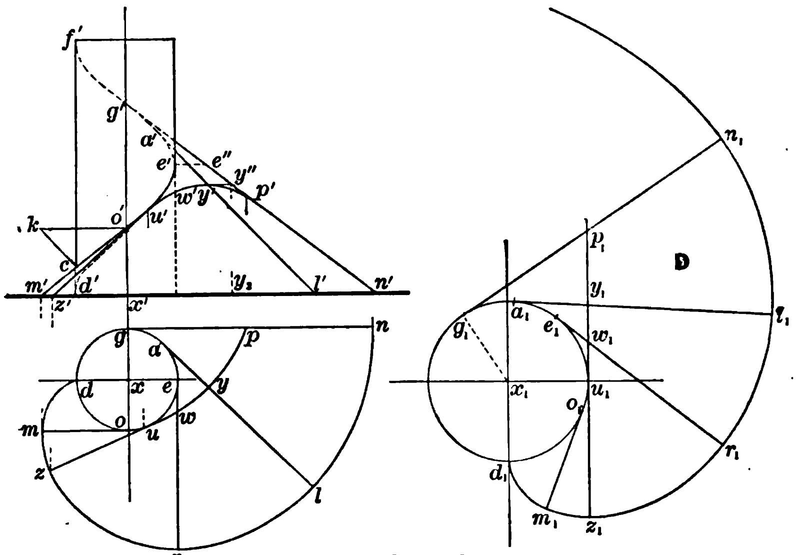


Fig. 168

between the point D and the element GN. The radius of curvature of the helix is $\frac{\rho}{\cos^2 \omega}$, in which ρ is the radius of the cylinder on which the curve lies, and ω is the *obliquity*, or angle made by the tangent with a plane perpendicular to the axis. This may be readily found graphically thus. The vertical projection of the tangent at O is $o'm'$, which cuts $d'f'$, the outline of the cylinder, at c; draw at o' a horizontal line, and at c a perpendicular to $o'm'$: these intersect at k, and $o'k$ is the radius of curvature required.

Then in the diagram **D**, draw about any centre x_1 a circle with this radius, and on it set off the arc $d_1o_1g_1$ equal to $g'n'$; this will be the development of the helical arc DOG. Since the horizontal trace drn is the involute of the original helix, its development will be the involute $d_1r_1n_1$ of the arc $d_1o_1g_1$, to which g_1n_1, equal to $g'n'$, will be tangent.

195. The Problem of the Shortest Path. Let it be required to find the shortest path on the surface, between the points D in **H**, and P on GN. These points fall in the development at d_1 and p_1, but they cannot be joined by a right line on the surface, since the latter has no existence within the circle. The least distance is found by drawing from p_1 a tangent to the circle, and finding the point of tangency u_1; the required line is, then, made up of the circular arc d_1u_1, and the right line u_1p_1.

The projections of this path on the original surface may be determined as follows. In the horizontal projection bisect the quadrant ge at a, and draw al tangent to the circle; it represents an element of which a_1l_1 is the developed position, where a_1 bisects the arc g_1e_1; this cuts p_1u_1 at y_1. The vertical projection of this point must lie on $a'l'$; in order to fix its altitude, set off $n'y'' = l_1y_1$, and draw through y'' a horizontal line cutting $a'l'$ in y'. Project y'' on **AB** at y_2; then $n'y_2$ will be equal to ly, the horizontal projection of l_2y_2 in its original position. In like manner, the location of W on ER may be found, and by drawing intermediate elements any desired number of points may be determined. The restored position of u_1 is best found by setting off ou, the same fraction of the quadrant oe that o_1u_1 is of the arc o_1e_1, and the helical arc DOU forms the first portion of the required shortest path.

The tangent to the helix at U is the element UZ, which in the development is u_1z_1, a prolongation of p_1u_1; and it will be noted that the shortest paths from P to any points on the portion DMZ of the horizontal trace of the convolute, are equal.

196. PROBLEM 6. *To find the intersection of a plane with any surface of revolution.*

Construction. Every transverse section is a circle, which in general is the simplest line that can be drawn upon such a surface; and these circles are made use of precisely as in the case of the

cone. Thus in Fig. 169, n' in the front view represents a chord in the circle whose radius is cd, which chord is seen in its true length as nn in the top view and as n_1n_1 in the supplementary view

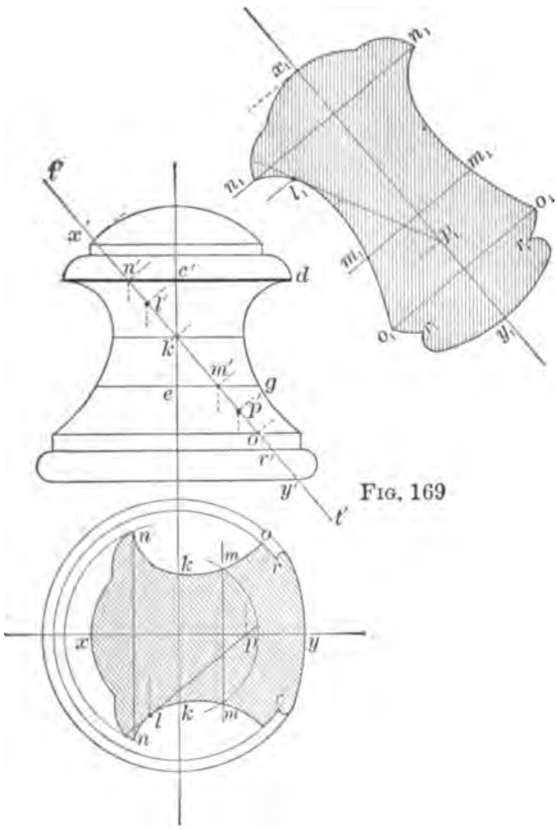


FIG. 169

S; m' represents a chord in the circle whose radius is eg, seen in its true length as mm and m_1m_1, and so on.

To draw a tangent to the curve of intersection, at any point as L. Draw the tangent to the horizontal projection at l, by the method of Fig. 139. This tangent pierces the plane through the axis parallel to V, at the point P; of which the supplementary projection is p_1: and l_1p_1 is the supplementary projection of the tangent.

Otherwise: Draw a plane tangent to the surface at L by the method of Fig. 153; its intersection with the given plane is the

required tangent line, whose supplementary projection may be found as in Figs. 163 and 167.

197. In Fig. 170 is shown the "stub end" of a connecting-rod. It is rectangular in section, and is joined to the cylindrical neck by a surface of revolution whose contour is the circular arc $w'z'$, described about the centre K: we have, then, to find the intersections of this surface by the two planes tt, ss, parallel to the axis.

A transverse section at c' is a circle whose radius is $c'p'$; this circle is seen in the end view to be cut by the plane tt at d, which is projected back to d', a point in the required curve; and in like manner other points may be determined.

The plane ss is seen in the side view to cut the outline at x', which determines the vertex x'' of the curve seen in the top view. A circle through e is seen in the end view to cut the vertical centre line in g, which, projected back to the contour at g', fixes the location of a transverse section from which by the preceding process would be found the point e' in the side view, corresponding to e'' in the top view. A transverse section at any point as o', between x' and g', is a circle in which nn, in the end view, is a chord; this chord is also seen in its true length as $n''n''$ in the top view. The same circle is seen in the end view to be cut by the plane tt at r, which projected back to r' determines another point in the curve of intersection in the side view. Other points may be found in a similar manner.

198. Tangents to the Curves of Intersection. By applying the method of (**175**), a tangent may be drawn to either of these curves at any point, with one exception.

It is seen in the end view that the circle through w is tangent to the plane tt, at the point y; and the intersection of this plane with the surface, as seen in the side view, consists of two symmetrical branches, which **intersect** at y'. The tangent plane at this point coincides with the cutting plane; consequently the tangent line cannot be determined by the usual process. But a tangent at this point to the lower branch may be constructed as follows:

The given centre of curvature at w' is the point K; produce Kw' to I, bisect KI at O, and about O describe a semicircle on KI as a diameter, cutting the projection of the axis at F. On KI set

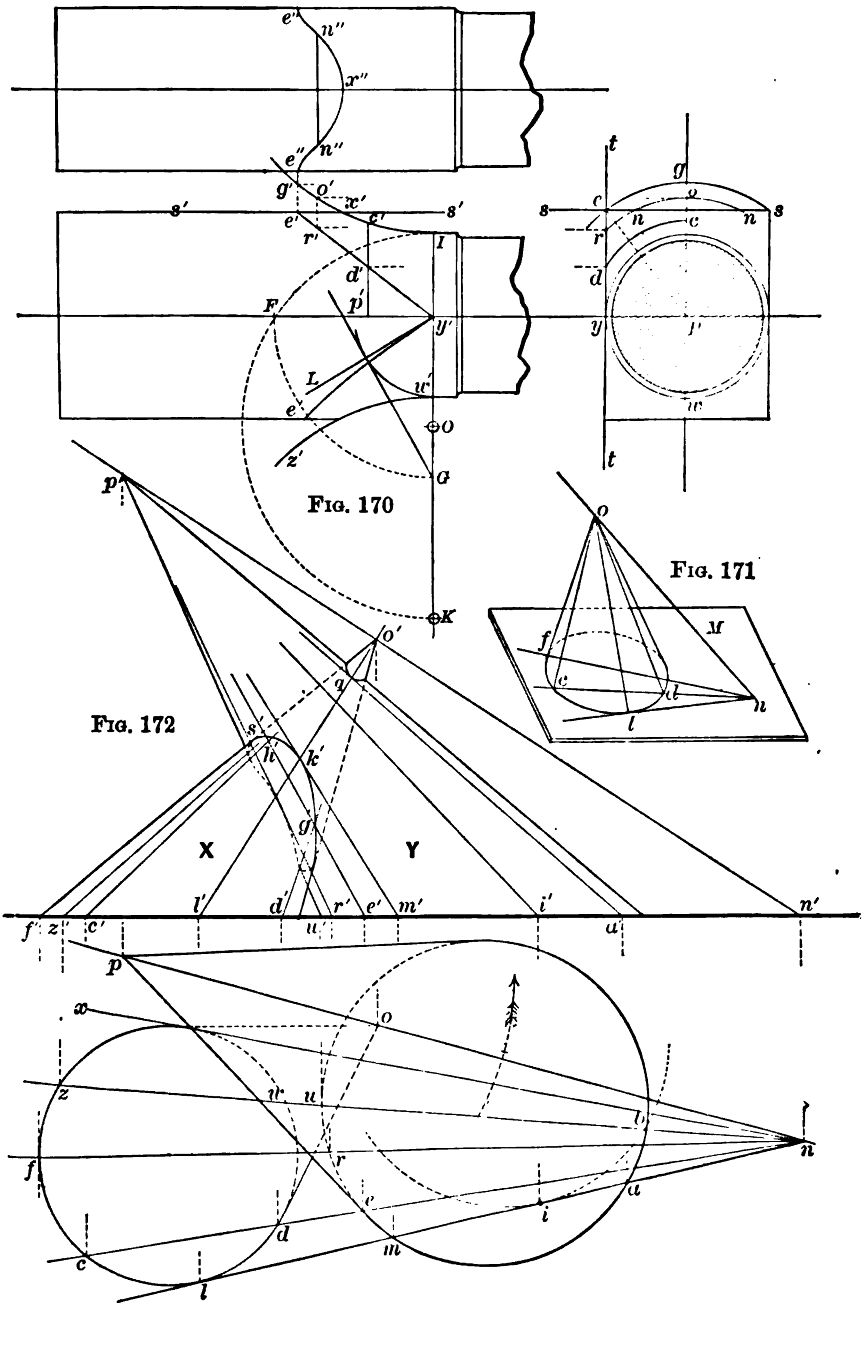


FIG. 170

FIG. 171

FIG. 172

off $y'G = yF$; also about y' as a centre describe a circular arc with radius $y'w'$: from G draw a tangent to this arc, then $y'L$, perpendicular to that line, is the required tangent at y'.

This construction is based upon the consideration that if the contour, $w'z'$, be the osculating circle of an hyperbola of which w' is the vertex and y' the centre, then $y'L$ thus determined will be one of the asymptotes. Had that hyperbola been the actual contour, the section of the surface by the plane tt, as will subsequently be shown, would have been, not a curve, but the right line $y'L$ itself.

INTERSECTIONS OF SINGLE-CURVED SURFACES.

199. In Fig. 171, draw any line on through the vertex o of the cone; it pierces the plane of the base, M, at the point n. Draw in the plane M any line nc through the point n, cutting the base at c and d; it is clear that the plane onc cuts from the cone the two elements oc, od. Draw in the plane M, through n, a line tangent to the base at l; then the plane onl is tangent to the cone along the element ol. And the like will be true of any other cone whose base is in the same plane M, if its vertex also lies in the same line on. It is now proposed to apply this in the solution of the following problem.

200. Problem 1. *To find the intersection of two cones whose bases are in the same plane.*

Analysis. Pass a series of planes through both vertices. These will cut elements from each cone; and the intersections of those cut from one cone with those cut from the other will be points in the required curve.

Construction. Since the bases are in the same plane, the two cones may be so placed that this plane shall coincide with **H**, as in Fig. 172. Join the vertices, P and O, by a right line, and produce it to pierce **H** in N. Then as in Fig. 171, nc is the horizontal trace of a plane which contains the line PN, and this plane cuts from the cone X two elements whose vertical projections are $c'o'$ and $d'o'$. The trace nc also cuts the base of the cone Y in e, vertically projected at e' in **AB**; and $e'p'$, the vertical projection of the element cut from that cone, intersects $c'o'$ and $d'o'$ in the points h', g', which therefore lie on the vertical projection of the required

curve: in a similar manner any desired number of points may be determined.

201. In the construction of such curves, there are certain criti cal and limiting points which it is always desirable to locate. For instance, $o'f'$ is the extreme visible element on the left, in the vertical projection of the cone X. And, just as in Fig. 171, fn is the horizontal trace of a plane containing that element and the line ON; this plane cuts from the cone Y an element whose vertical projection $r'p'$ intersects $o'f'$ in a point s': and at this point $o'f'$ is tangent to the vertical projection of the curve. The points of contact on the right-hand element of X, and on the left-hand element of Y, are found in a similar manner.

Draw nl tangent to the base of X; it is the horizontal trace of a plane tangent to that cone along the element whose vertical projection is $l'o'$; this plane cuts from Y an element whose vertical projection $m'p'$ is tangent to the vertical projection of the curve at k', its intersection with $l'o'$. And another tangent may be determined by means of another tangent plane, whose horizontal trace is nx.

202. Attention has thus far been purposely confined to the vertical projection of the curve. The horizontal projection, of course, will be determined by the intersections of the horizontal projections of the elements; but the effect of introducing these, in a diagram upon so small a scale, and necessarily involving so many other lines, would have been simply bewildering, for which reason they have been omitted. In constructing that projection, it will be found advantageous to pass auxiliary planes through the extreme visible elements of the cones in the top view also, in order to locate the points at which the curve appears tangent to those elements.

It is also to be noted, that the two projections of these curves are in a sense independent of each other. Thus, the bases, and the vertical projections of the cones, remaining as they are, the horizontal projections p, o, of the vertices may lie upon *any* oblique line drawn through n. Whence it appears that an infinite number of pairs of cones may be constructed having the same bases and altitudes, whose curves of intersection will all have the same verti-

cal projection. In a similar manner it may be shown that different pairs of cones may intersect in curves which have the same horizontal projection.

203. Fig. 172 represents a case of complete interpenetration; the cone X enters the cone Y on the left, passes bodily through it, and emerges on the right, thus forming two distinct curves of intersection. This is indicated in the horizontal projection by the circumstance that every one of the horizontal traces, including the two which are tangent to the base of X, cuts the base of Y in two points. Each of the auxiliary planes, then, cuts two elements from Y, although it may contain only one element of X; but the latter will intersect both the former; thus, $l'o'$ cuts $m'p'$ at k', and it also cuts $a'p'$ at q', which is the vertical projection of the point of tangency between the second element and the other curve.

204. Now suppose the cone Y to be revolved around a vertical line through P, in the direction indicated by the arrow. The points a and m, and consequently the points q' and k', will approach each other, until when the base becomes tangent to ln as shown by the dotted line, the auxiliary plane will be tangent to both cones. The two curves will then coalesce, forming one continuous line, which, as shown in Fig. 173, will *cross itself* at the intersection of LO and IP, the two elements of contact, not being tangent to either of them.

The interpenetration is still complete; but the exact limit has now been reached, and if the revolution of Y be carried any farther, this will no longer be the case. The plane tangent to X along the element LO will then pass outside of the cone Y; no part of either surface will be entirely buried within the other, and the two will intersect, as shown in Fig. 174, in a continuous curve which does not cross itself. In this diagram the cones are of equal altitude; the line joining the vertices is therefore parallel to **H**, and for convenience it is here made parallel to **V** also, so that the traces of the auxiliary planes are parallel to **AB**. The auxiliary plane tangent to Y along the nearest element IP, cuts from X the two elements JO, LO, which are tangent to the curve at the points C, D, in which they intersect IP. Similarly, an auxiliary plane tangent to X along its most remote element, will cut from Y two ele-

ments, both of which are tangent to the curve; and other points are found as previously explained.

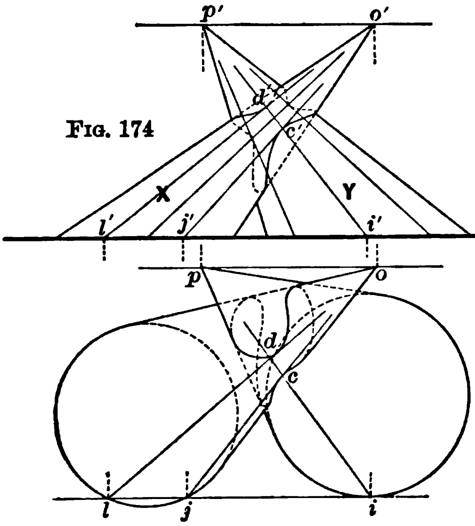


FIG. 174

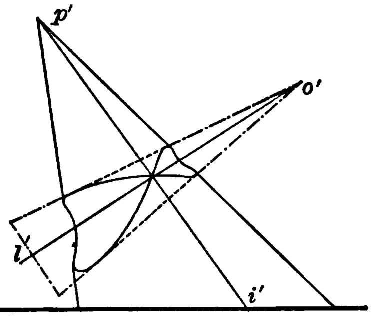


FIG. 173

205. If the cones as given have not a common base, a very obvious expedient is to provide one, by passing a plane which cuts all the rectilinear elements of both. This, however, is not absolutely necessary nor always even desirable. The base of Y, Fig. 175, being in the horizontal plane, it is clearly always possible so to place the two cones that the base of X shall lie, as here shown, in a plane tTt', perpendicular to **V**. Drawing OP, produce it to pierce **H** in N, and the other plane in M. Revolve tTt' into **V**; then the base of X will be seen in its true form and in its correct position relatively to the point M, which falls at m''. All the traces of the auxiliary planes upon tTt', then, will pass through m'', and all their horizontal traces through n. Draw $m''f''$, tangent to the base at e''; this is the trace of a plane tangent to X along the element EO. Set off $Tf = Tf''$, and draw nf; this is the horizontal trace of the same plane, which is now perceived to cut from Y the two elements CP, DP: these cut EO in two points of the curves sought.

The construction, in short, is effected precisely as in the preceding cases: for instance, in order to find a point on RP the extreme right-hand element of Y, draw nrg the horizontal trace of an auxiliary plane containing that element, set off $Tg'' = Tg$, and

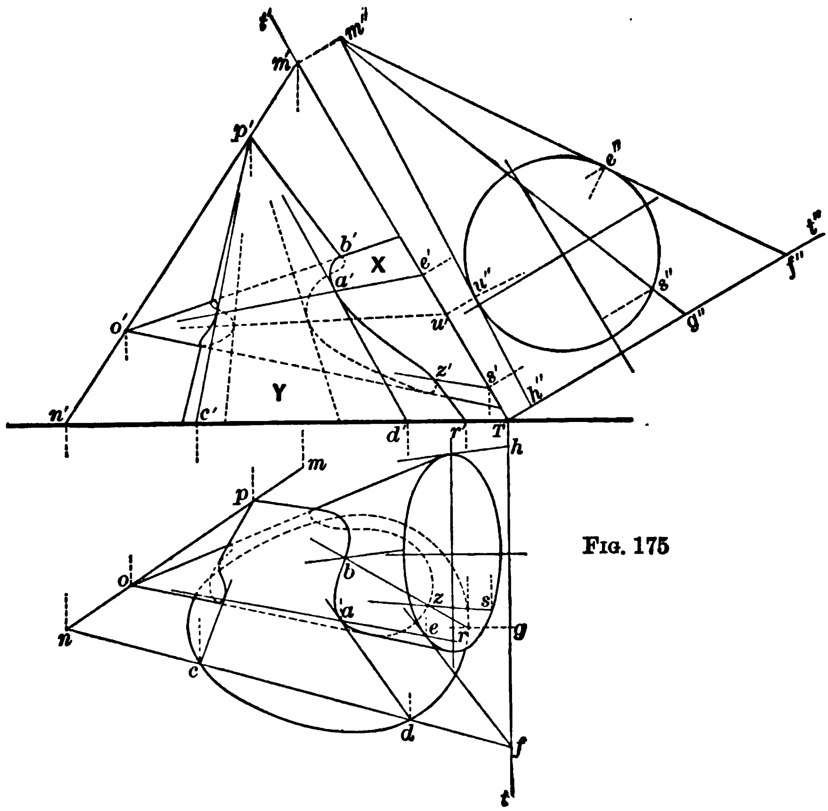


FIG. 175

draw $g''m''$ cutting the base of X at s''; project s'' to s', then $s'o'$ is the vertical projection of an element, cutting $r'p'$ in z', which is the vertical projection of the required point.

206. Manipulation in Construction.—It is a very common mistake to suppose that in making such constructions as those of Figs. 172 and 175, time may be saved by drawing at once both projections of a great number of elements, and then selecting those which intersect in the required points. But the effect of such a maze of lines is bewildering, and the attempt is more than likely

to result in error, confusion, and absolute loss of time. A much safer and on many accounts better course is to find first a point situated for instance on the nearest element of one of the surfaces, and then to follow the curve around point by point in one direction, sketching it in lightly as each additional point is located. In determining a point, do not draw in the trace of the auxiliary plane, but simply mark where it cuts the bases; and do not draw the whole projection of either element, because ordinarily a short bit of *one* will suffice, upon which is marked the point in which the other cuts it.

207. If the cones are so situated as to have two auxiliary tangent planes in common, as in Fig. 176, then it is clear that all the

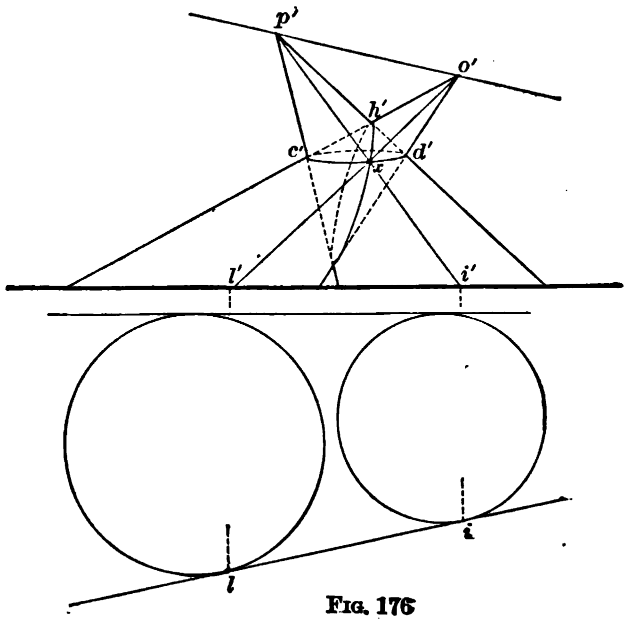


FIG. 176

elements of both surfaces will be cut, and that the line of intersection will cross itself at two points on opposite sides of the cones. The line may be of double curvature; but **if the bases of both cones are conic sections,** it will in general be composed of **two plane curves,** which therefore are themselves conic sections.

An exceptional case is illustrated in Fig. 177, where p, o, are

the vertices of two similar cones of revolution of equal altitude, whose axes are vertical and therefore parallel.

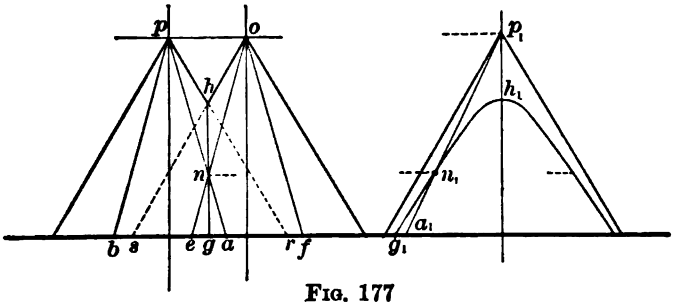


Fig. 177

The extreme elements *pr*, *os*, intersect in *h*; an auxiliary plane through *po* cuts from one cone the elements *pa*, *pb*, and from the other, the elements *oe*, *of*: *pa* cuts *oe* in *n*, but it is parallel to *of*, and *oe* is parallel to *pb*. Consequently these lower nappes of the cones can intersect in only *one* line, the hyperbola *hng*; the upper nappes obviously intersect in the opposite branch of the same curve: and it is quite evident that a similar state of things may exist with cones of other forms.

In Fig. 178, the angles at the vertices are the same as in Fig. 177, and the vertices are equidistant from *g*, the intersection of the

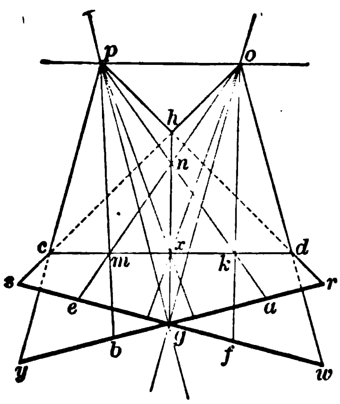


Fig. 178

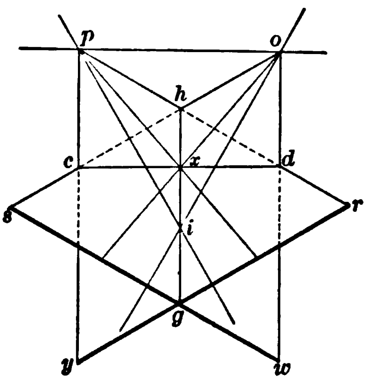


Fig. 179

axes, whose inclination is such that the extreme elements *py*, *ow*, are not parallel. In this case *pa* not only cuts *oe* in *n*, but it cuts

of in *k*, and *oe* cuts *pb* in *m*; thus the lower nappes intersect in the hyperbola *hng*, and also in the ellipse seen edgewise as the right line *cd*: the upper nappes intersect in the opposite branch of the same hyperbola.

In Fig. 179 the same cones are shown, the vertices equidistant from *i* the intersection of the axes, but so inclined that *py* and *ow* are parallel. In these circumstances the surfaces intersect in the parabola *hig*, and also in the ellipse *cd*; the upper nappes do not meet at all. The same cones, it is clear, can be so placed as to intersect each other in two ellipses, as do those shown in Fig. 176: in which case also the opposite nappes do not intersect.

208. In the case of any two given cones, the question whether there will be any line of intersection with infinite branches may be decided thus: Draw through the vertex of either a series of lines parallel to the elements of the other; then in general this third cone will either have only the vertex in common with the first, or be tangent to it along one element, or cut it in two elements.

In the first case, no element of either of the given cones will be parallel to any element of the other, and the intersection will consist of one or two closed curves; which, whether of double curvature or not, belong by analogy to the class of *elliptic intersections.*

In the second case, one element of each cone will be parallel to one element of the other; and there will be one line of intersection consisting of an infinite branch with no asymptote: which therefore belongs to the class of *parabolic intersections.*

In the third case, two elements of each cone will be parallel to elements of the other; the surfaces will intersect in two infinite branches with asymptotes, which belong by analogy to the class of *hyperbolic intersections:* and the asymptotes are determined by drawing planes tangent to the cones along the parallel elements; these will intersect in the required lines.

In the last two cases there will be one closed curve of intersection, besides the infinite ones. But should the two cones have all their elements respectively parallel, as in Fig. 177, there will be an intersection composed of two infinite branches, and no other. The

cones in this case have necessarily two common tangent planes, and the elements of contact are the ones to which the asymptotes will be parallel. If the intersection is of single curvature, the asymptotes will be determined by the intersections of the plane of the curve with the two tangent planes; but if not, the asymptote cannot be graphically determined: because the planes tangent to the two cones along the parallel elements now coincide, giving no line of intersection.

209. If one of the cones becomes a cylinder, the intersection may be found in substantially the same manner. Draw a line, parallel to the elements of the cylinder, through the vertex of the cone; then auxiliary planes through this line will cut from both surfaces rectilinear elements, whose intersections will be points in the required curve. Draw such planes tangent to each surface; then, as before, if both planes tangent to either cut the other, there will be two separate curves; if one plane tangent to each cut the other, there will be one continuous curve; if there be one common tangent plane the curve will cross itself once; and if two, it will cross itself twice, and will consist of two conic sections when the base of each surface is a conic section.

Infinite Intersections.—Since all the elements of the cylinder are parallel to the **auxiliary line** through the vertex, they cannot be parallel to any element of the cone, unless that line itself lies in the conical surface; if it does, there may be curves of intersection with infinite branches. Draw a test plane tangent to the cone along this auxiliary line; it will either pass outside the cylinder, or be tangent to it, or cut it in two elements.

In the first case, the intersection will be a single closed curve.

In the second case, it will in general consist of one infinite branch with a single asymptote.

In the third case it will in general consist of two infinite branches with two common asymptotes.

210. That these things are so, is very clearly shown by so placing the surfaces that the auxiliary line is vertical. Thus in Fig. 180, which is a horizontal projection, let o represent this line; let M be the horizontal trace of the cone, N that of the cylinder, TT that of the test plane, and ox, oy, oz, those of **auxiliary** planes.

Each auxiliary plane contains the auxiliary line, which is parallel to the elements of the cylinder; but it also cuts from the cone another element which is *not* parallel to them, and since M is in the horizontal plane, while the vertex is above it, it is apparent that in this

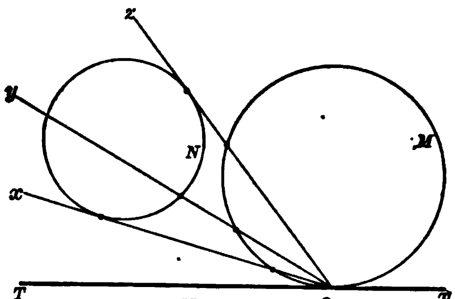


FIG. 180

case every element of the cylinder must pierce the cone in a point of the lower nappe. Had the cylinder been placed on the opposite side of TT, the intersection would have lain on the upper nappe.

Fig. 181 illustrates the second case, the test plane being tangent

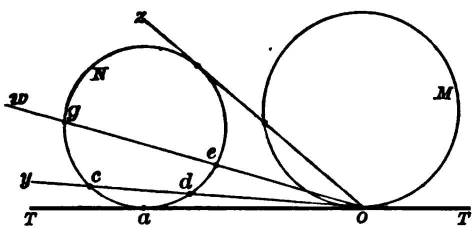


FIG. 181

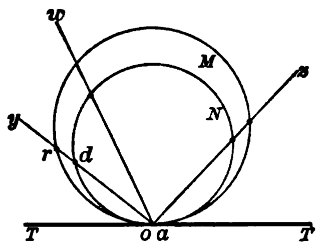


FIG. 182

to N; here it is evident that as the auxiliary plane approaches coincidence with TT, the elements cut from the cylinder approach the line of contact a, while the one cut from the cone becoming more nearly parallel to them, meets them at points more and more remote, receding to infinity at the limit. Consequently the curve of intersection lies wholly on the lower nappe, and the element a of the cylinder is an asymptote to both sides of the single infinite branch, and lies between them, but nearer to one than to the other.

An exception to this occurs when, as in Fig. 182, the two surfaces are tangent along an element. Let oy be the trace of any

auxiliary plane, cutting M in r and N in d: then or is the projection of that part of an element of the cone included between the vertex and the base, and od the projection of that part included between the vertex and the line of intersection. The length of the former being always finite, that of the latter must also be finite; therefore the surfaces intersect in a single closed curve.

211. The third case is illustrated in Fig. 183; by comparison with Fig. 181, it is obvious that the part of the cylinder included between TT and the tangent auxiliary plane oz, will intersect the cone in an infinite branch on the lower nappe, to which the two

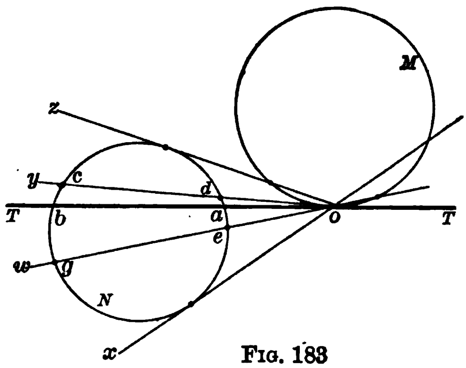


FIG. 183

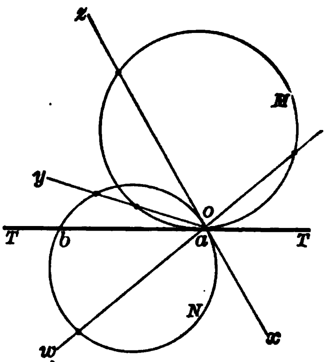


FIG. 184

elements a and b of the cylinder will be asymptotes. Also that the portion on the opposite side of the test plane will give another infinite branch on the upper nappe, having as asymptotes the same elements a and b.

There will be an exception to this also, when the two surfaces intersect in the auxiliary line, as in Fig. 184. They will also intersect in one continuous curve, which, crossing the common element at the vertex, extends both ways to infinity, and is asymptotic in each direction to the other element b cut from the cylinder by the test plane.

212. In the preceding arguments relating to the nature and the number of the curves of intersection, it has been assumed that, as is usual in practical cases, the cones and cylinders have closed curves as bases, and are externally convex throughout. Almost endless variations might result from constructing one or both of the

surfaces with spiral, sinuous, or infinite curves for bases; these however it is not proposed to consider, since they involve no principle which has not already been explained.

213. Intersection of Two Cylinders. The same general method is still applicable: Draw a plane parallel to the elements of each cylinder; then auxiliary planes parallel to this plane will cut from both surfaces rectilinear elements, and their intersections will be points in the required curve. By drawing two such planes tangent to each surface, the questions as to whether there will be one or two curves, one or two crossings, etc., are decided exactly as in (**208**); and when there are two common tangent planes, the intersection will consist of two conic sections, if the bases themselves are conic sections.

As to infinite intersections, it is perfectly obvious that, if the bases are closed curves, there can be none, unless the elements of one cylinder are parallel to those of the other; when they will intersect in either two or four elements, if at all. But any infinite curve can be made the common base, and therefore the line of intersection, of two cylinders.

214. Other Methods of Operation. It appears then from the foregoing, that if two cones, two cylinders, or a cone and a cylinder, intersect each other, it is always *possible* to employ a system of auxiliary planes which cut rectilinear elements from each surface. But it must not be inferred that this is the only or always even the most eligible expedient. In probably the greater number of practical cases, these surfaces have circular bases; and when this is so, planes which cut circles from each, or right lines from one and circles from the other, can often be employed to great advantage.

Thus in Fig. 185, the horizontal plane $i'f'$ cuts from the cone a circle, which in the horizontal projection is seen to pierce the cylinder at e and n, which are vertically projected to e', n'; and any number of points may be found in like manner.

Draw ox, oy, tangent to the base of the cylinder; describe through r and c, the points of contact, a circle about o, cutting ow in d, whose vertical projection is d'; then r', c', must lie on a horizontal plane through d', and at those points the curve is tangent to $o'x'$, $o'y'$. Draw through a, the centre of the base of the cyl-

inder, the right line *ov*; it cuts the base in *s* and *k*, whose vertical projections, found in the same manner as *r′* and *c′*, are the highest and lowest points of the intersection.

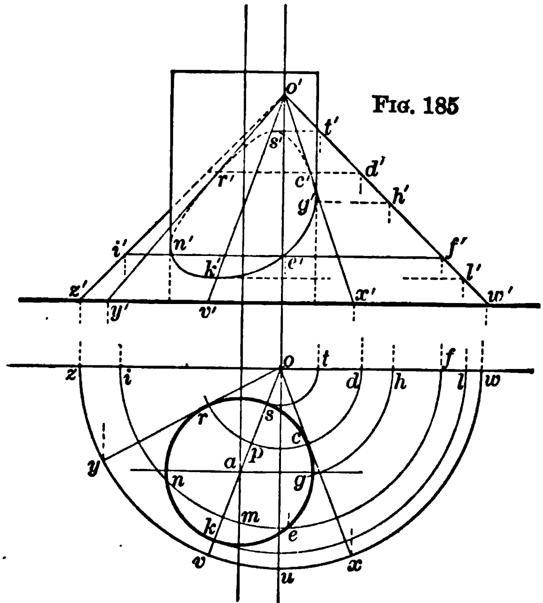


FIG. 185

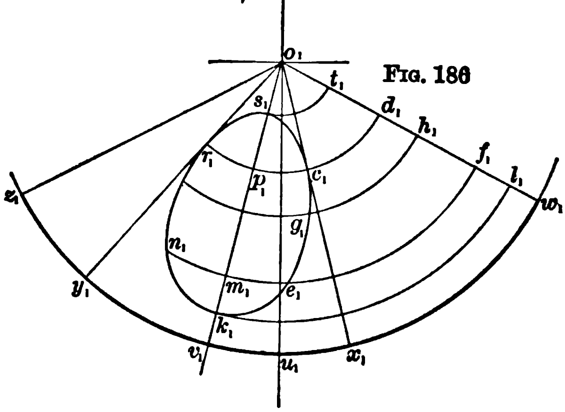


FIG. 186

215. Only the front half of the cone is represented in Fig. 185; and in Fig. 186 is given the development of that half, showing the form of the hole which must be cut in the sheet for the insertion of the cylinder. The base develops into the arc of a circle whose radius o_1z_1 is equal to $o'z'$, the slant height of the cone, and the

length of the arc $z_1u_1w_1$ is equal to that of the semicircumference zuw. Make the arc u_1v_1 = arc uv, set off the arcs v_1x_1, v_1y_1, equal to the arc vx, and draw o_1y_1, o_1v_1, o_1x_1. Describe about o_1 an arc with radius o_1d_1, equal to $o'd'$, thus locating the points of tangency r_1, c_1; describe about o_1 an arc with radius $o_1f_1 = o'f'$, cutting o_1v_1 in m_1, set off the arcs m_1n_1, m_1e_1, each equal to me, then e_1, n_1, lie on the developed curve: and other points may be found in a similar manner. The vertices, s_1 and k_1, are determined by setting off on o_1v_1 the true distances of the points S and K from the vertex of the cone, which are respectively equal to $o't'$, $o'l'$.

216. The construction of Fig. 187 may be explained thus: The point in which any line on either the horizontal or the inclined cylinder pierces the vertical one, is seen directly in the top view. Thus, the nearest element hk of the horizontal cylinder cuts the circumference at k, which is projected vertically to k' in the front view: any point c'' on the circumference in the end view represents an element, seen as $c'd'$ in the front view, and as cd in the top view, where the distance xc from the centre line is equal to $x''c''$; and d is vertically over d. Similarly, the nearest element of the inclined cylinder is represented by $m'o'$ in the front view and by mo in the top view, and the altitude of the point in which it pierces the upright cylinder is determined by projecting o up to o'. A supplementary view looking in the direction of the arrow shows the base of the inclined cylinder in its true form, and any point n_1 upon its circumference represents an element of which the front view is $n'r'$ and the top view is nr, located by making $wn = w_1n_1$; and r' is vertically over r: any number of points in the curves may be found in the same manner. In this way the determination of the intersections might be made clear to one totally unfamiliar with the stage machinery of Descriptive Geometry: though it is very evident that the operations are actually equivalent to the use of a series of horizontal planes in the first case and a series of vertical ones in the second.

217. Just such cases as these last are the ones most often met with in practice; and accordingly, they are the ones most seldom illustrated in theoretical treatises on the principles involved in them. In the application to sheet-metal work, the development is of im-

portance; and that of the upright cylinder is given in Fig. 188. Supposing it to be cut vertically along the most remote element u, and unrolled to right and left, the surface will form a sheet of a breadth equal to the height of the cylinder, and a length equal to its circumference. The intersection with the horizontal cylinder

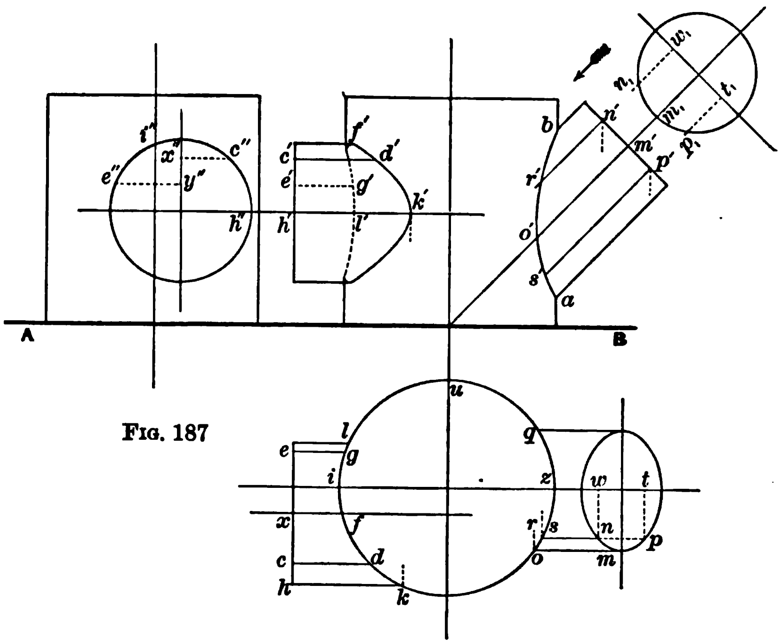


Fig. 187

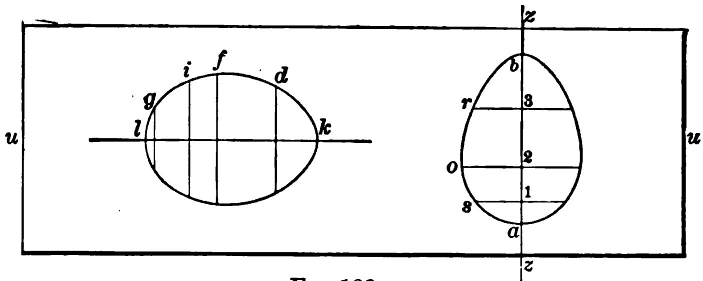


Fig. 188

will develop into a curve symmetrical about a horizontal line whose distance from the lower edge is equal to that of the axis from **AB** in Fig. 187. Rectify the arcs ul, lk, in the top view, in the development set off the distances ul, lk, equal to them, and subdivide the latter into parts respectively equal to the partial arcs kd, df,

etc.; at the points of subdivision erect vertical ordinates equal to the distances of d', f', etc., above the centre line of the horizontal cylinder in the front view; the required curve passes through the extremities of these ordinates. The intersection with the inclined cylinder will develop into a curve symmetrical about zz, the position of the right-hand element of the upright cylinder on the unrolled sheet; the altitudes of the vertices, a, b, are taken directly from the front view: on zz mark also the points 1, 2, 3, at altitudes equal to the distances of s', o', r', above **AB**, and at these points draw horizontal ordinates $1s$, $2o$, $3r$, respectively equal in length to the rectified arcs sz, oz, rz, in the top view: any desired number of points may be found in a similar manner.

INTERSECTIONS OF DOUBLE-CURVED SURFACES.

218. PROBLEM 1. *To find the intersection of two surfaces of revolution whose axes are in the same plane.*

Analysis. If the axes intersect, take the point of intersection as the common centre of a series of auxiliary spheres. Each sphere will cut a circle from each of the given surfaces (**237**); the circumferences of these circles will cut each other in two points, which lie upon the required curve. If the axes are parallel, the spheres become planes perpendicular to the axes.

Construction. In the side view at the left, Fig. 189, the plane of the axes, which is parallel to the paper, contains the visible con-

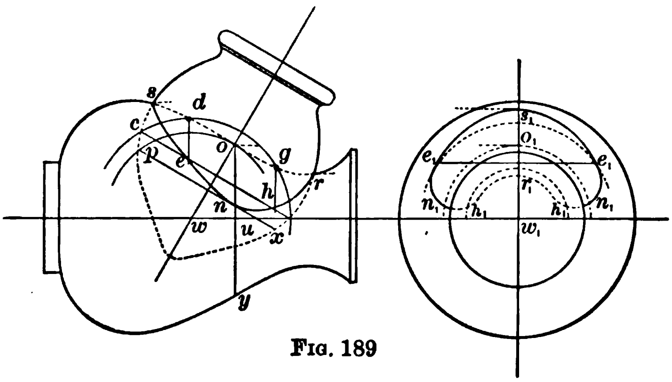


FIG. 189

tours of the given surfaces and of the spheres whose common

centre is w; the intersections of the former, at r and s, give at once two points of the required curve. A sphere tangent to one surface around the circle oy, cuts the other in the circle px, and oy cuts px in n; another and larger sphere cuts the first surface in circles through d and g, and the second in a circle through c, and the intersections of these circles at e and h also lie upon the required curve snr: any desired number of points may be determined in like manner.

In the end view, the points s, r, are projected directly to s_1, r_1, on the vertical centre line. The point n in the side view represents a chord in the circle oy, whose circumference being seen in its true form and size in the end view, the points n_1, n_1, are found by projecting n across to that circumference; and e_1, h_1, are located in a similar manner.

Note. This method is equally applicable to the case of single-curved surfaces of revolution whose axes intersect. Its application in the extreme case where the intersection is infinitely remote and the spheres become planes, has already been illustrated in finding the intersection of a cone with a cylinder, Fig. 185.

219. If the axes lie in different planes the determination of the intersection of the surfaces is in general much more laborious. In most cases, it would probably be advisable to use a system of auxiliary planes perpendicular to one of the axes, thus cutting circles from the surface to which that axis belongs; but it still remains to determine the form of the line cut from the other surface by each individual plane.

220. PROBLEM 2. *To find the intersection of any oblique cone with a sphere.*

Analysis. Pass a series of auxiliary planes through the vertex; each will cut a circle from the sphere and two elements from the cone; and the intersections of these elements with the circumference of that circle will be points in the required curve.

Construction. In Fig. 190, C is the centre of the sphere, O is the vertex of the cone, whose base for the sake of convenience is placed in **H**; and the auxiliary planes are vertical. Let pp, drawn at pleasure through o, be the horizontal trace of one of these planes, cutting the base of the cone in d and r, and the contour of the

sphere in f and h; then fh is the horizontal projection of the circle cut from the sphere, and its middle point e is that of its centre. Revolve this plane about pp into **H**; o goes to o'', oo'' being equal to the altitude of the cone; and do'', ro'', are the revolved positions of the elements cut from the cone. Also, e goes to e'', the distance ee'' being equal to that of C from **H**; about e'' with radius ef describe an arc cutting do'', ro'', in g'', u'', which will be the

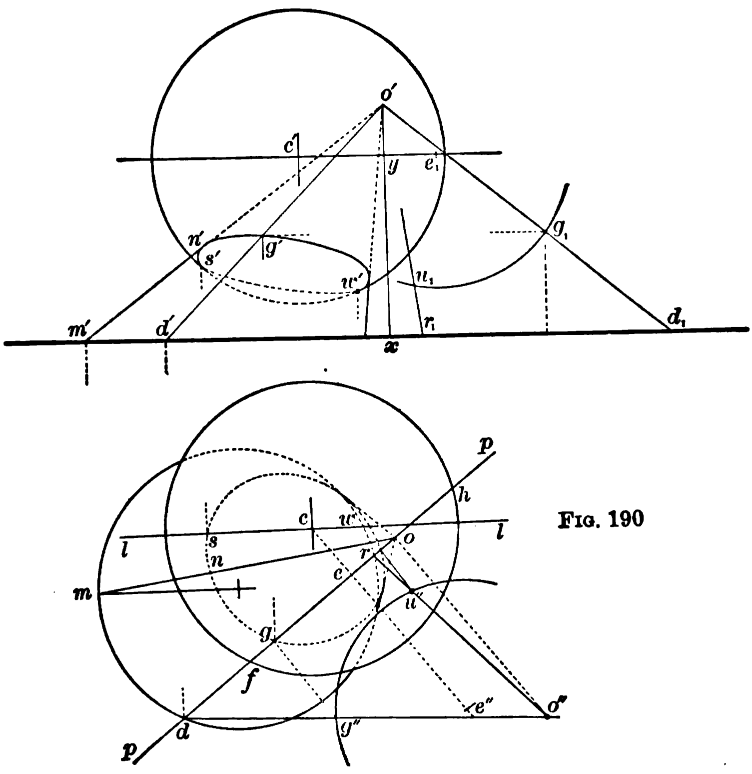


FIG. 190

revolved positions of the points in which the elements pierce the sphere. In the counter-revolution, g'' goes to g on pp, whence it is vertically projected to g' on $d'o'$, thus determining a point G on the required curve; in like manner the projections of the point whose revolved position is u'' may be found, the construction being omitted to avoid confusion in the diagram.

Otherwise: Revolve the auxiliary plane about the horizontal

projecting line of O until it is parallel to **V**; this line is $o'x$ in the vertical projection, and setting off $xr_1 = or$, and $xd_1 = od$, the two elements appear as $o'r_1$, $o'd_1$: on the horizontal through c' set off $ye_1 = oe$, then e_1 is the revolved position of the centre about which an arc with radius ef is to be described, cutting the elements in u_1 and g_1.

221. By repeating the above process, any number of points may be found and the curve fully determined. The points at which the projections of this curve are tangent to the extreme visible elements of the cone, as for instance n' on $o'm'$, are determined as in previous cases, by passing auxiliary planes through those elements. But the points at which the vertical projection is tangent to the contour of the sphere cannot be located by any direct means, since there is no way of ascertaining which element of the cone will pierce the sphere in the great circle cut from it by the plane ll parallel to **V**. These points may, however, be determined indirectly, thus: the horizontal projection of the curve cuts ll at s and w; these are the horizontal projections of those points of penetration, and the vertical projections s' and w' must lie on the contour of the sphere.

222. In this instance, the vertex of the cone lies within the body of the sphere; when it lies outside the surface there may be two curves of intersection, but the construction is the same for both. If any plane tangent to the cone passes outside the sphere, the interpenetration will be partial, and the surfaces will intersect in one continuous curve; if the cone and sphere have one common tangent plane, this curve will cross itself once; if they have two it will cross itself twice. In order to ascertain whether either condition exists, draw a line from the centre of the sphere to the vertex of the cone, and make it the axis of a cone of revolution with the same vertex and tangent to the sphere; if this test cone is tangent to the given one along one element, there will be one common tangent plane; and if the cones are tangent along two elements, there will be two of them. If the cones cut each other in two elements, there will be one closed curve of intersection; if in four elements, the given cone will intersect the sphere in two distinct curves. It is here assumed, as in (**212**), that the transverse section of the

given cone by a plane perpendicular to the axis of the test cone, is a closed curve and externally convex throughout.

The intersection of a sphere with a cylinder, obviously, is to be determined by means of a system of planes cutting elements from the latter and circles from the former, the process of construction being substantially the same as in the case of the cone.

223. PROBLEM 3. *To develop any oblique cone.*

Analysis. Intersect the cone by a sphere whose centre is at the vertex. The curve of intersection will develop into an arc of a circle whose radius is equal to that of the sphere. On this circle lay off distances equal to the rectified arcs of the curve of intersection intercepted between rectilinear elements of the cone; and on the radii drawn through the points thus determined, set off the true lengths of the corresponding elements.

Construction. The intersection with the sphere, in Fig. 191, is constructed as in Fig. 190. Since it is a double curved line, its true length is not seen in either projection; but it can be ascertained by developing the horizontal projecting cylinder, as shown in $\mathbf{D}_1$. Here the elements of the cylinder appear in their true length and at their true distances from each other; thus e_1x_1 is equal to $e'x$, g_1y_1 to $g'y$, and x_1y_1 to the arc eg in the horizontal projection; the positions of other elements being determined in like manner, the base develops into a right line w_1w_1, and the double curve line into a single curved one $n_1e_1n_1$. In the development of the cone, shown in $\mathbf{D}_2$, an indefinite circular arc is described about any centre o_2 with a radius equal to that of the sphere; on this, set off an arc e_2g_2 equal to the arc e_1g_1 in $\mathbf{D}_1$, then $g_2m_2 = g_1m_1$, and so on: the points thus determined fix the positions of the elements of the cone on the developed sheet, and on the radii drawn through them the true lengths of these elements are set off,—as $o_2d_2 = OD$, $o_2f_2 = OF$, etc. In a similar manner the developed form of any other curve on the surface may be determined.

To draw a tangent to the developed base at any point, as f_2. The line o_2f_2 is the developed position of the element OF, and the tangent to the base of the cone at F is FQ, in the horizontal plane. Find the true angle included between OF and FQ, and make the angle $o_2f_2q_2$ equal to it; then f_2q_2 is the required tangent. And the

tangent to the development of any other curve on the surface may be drawn in the same way.

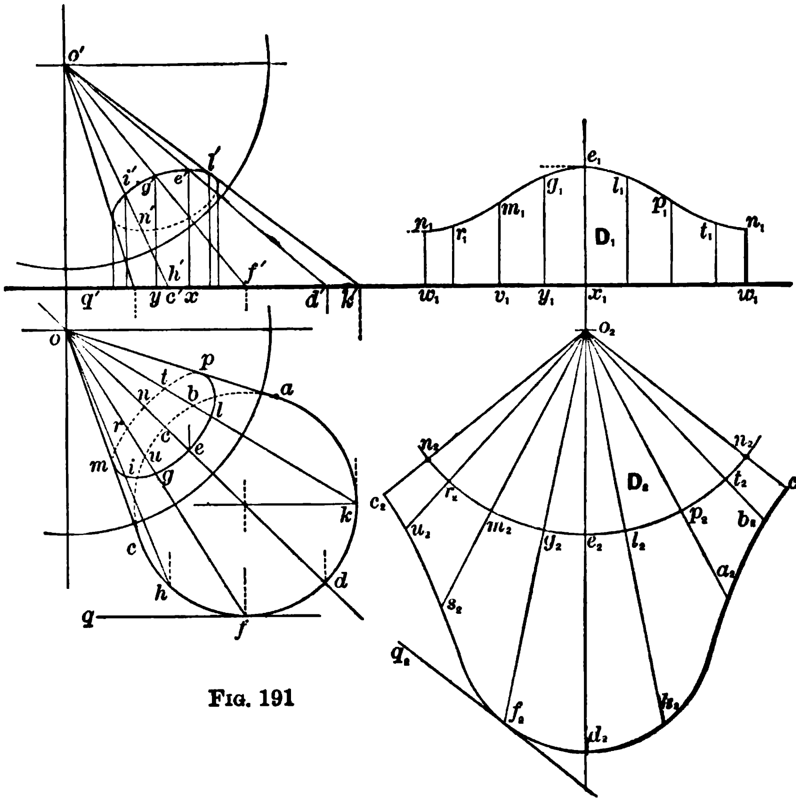


FIG. 191

224. Conditions of Symmetry. The cone represented in Fig. 191, having a circular base, is symmetrically divided by the vertical plane through the axis, which also cuts from it the longest element OD and the shortest one OC: and the points E, N, in which these elements pierce the sphere, are respectively the highest and the lowest points of the curve of intersection. Again, any two elements OF, OK, equidistant from OD, will be of equal length, make equal angles with the plane of the base, and therefore pierce the sphere in points of equal altitude; moreover, since they are equally foreshortened in the horizontal projection, *og* is equal to *ol*, and *eg* equal to *el*.

From these considerations it follows that the horizontal projection of the curve of intersection is symmetrical with reference to od; that the development of the projecting cylinder is symmetrical about e_1x_1; and that the development of the cone is symmetrical about o_1d_1. And it is evident that the same will hold true in regard to any cone which is symmetrically divided by a plane through the vertex perpendicular to the plane of the base.

225. Practical Suggestions. The above deductions are of importance in the practical applications of this problem. For, a curve which is symmetrical with respect to a given line can always be constructed more easily and more accurately than one which is not: and the process of re-forming the original cone from the developed sheet is likewise much facilitated if the latter be symmetrical.

Such work as this should in practice always be laid out upon as large a scale as may be; when this is done, the processes of rectifying the arcs of the horizontal projection of the curve of intersection, in order to develop the projecting cylinder, and of transferring the length of the developed curve to the circular arc in developing the cone, may be executed with sufficient accuracy by stepping them off with the spacing dividers, the points of which are set so close together that the difference between the chord and the arc shall be practically inappreciable.

CHAPTER VI.

Of Warped Surfaces.

The Hyperbolic Paraboloid; Its Vertex, Axis, Principal Diametric Planes and Gorge Lines. The Conoid.—The Hyperboloid of Revolution.—The Elliptical Hyperboloid, and its Analogy to the Hyperbolic Paraboloid. The Helicoid; of Uniform and of Varying Pitch. The Cylindroid.—The Cow's Horn.—Warped Surfaces of General Forms.—Planes Tangent to Warped Surfaces. Warped Surfaces Tangent to Each Other. Intersections of Warped Surfaces.

THE HYPERBOLIC PARABOLOID.

226. The Hyperbolic Paraboloid is a warped surface, with a plane directer, and two rectilinear directrices which lie in different planes. It takes its name from the fact that its curved sections by planes are either hyperbolas or parabolas.

Any plane parallel to the plane directer will cut each directrix in a point; and the right line joining these two points will be an element of the surface.

If a series of such parallel planes be drawn, they will divide the two directrices proportionally. And conversely: If any two right lines not in the same plane be divided into proportional parts, the right lines joining the corresponding points of division will lie in parallel planes, and be elements of a hyperbolic paraboloid.

If through any point in space two lines be drawn, respectively parallel to any two of these elements, the plane of those two lines will be parallel to the plane directer, and may be taken for it.

If in this plane directer any right line be drawn, the element parallel to that line may be found thus: Through any point of either directrix draw a line parallel to the given line; that directrix and this parallel will determine a plane cutting the other directrix in a point: through that point draw a parallel to the given line, and it will be the element required.

227. The Hyperbolic Paraboloid is Doubly Ruled; that is to say, it has two sets of rectilinear elements, and consequently two plane directers.

In Fig. 192, let XX be the plane directer, AC and BD the directrices, CD and AB two elements, the latter lying in XX.

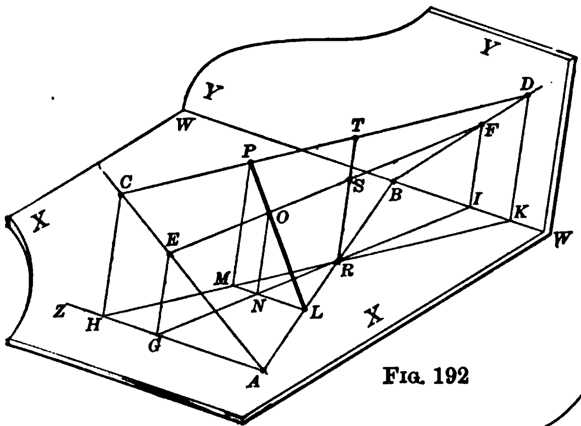


FIG. 192

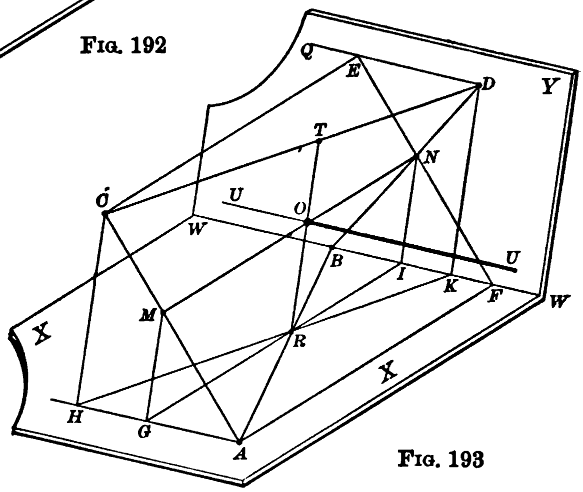


FIG. 193

Through BD draw a plane YY parallel to AC, cutting XX in WW: a plane parallel to this through AC cuts XX in AZ parallel to WW. Draw DK perpendicular to WW, also CH perpendicular to AZ; then HK will be parallel and equal to CD.

Through any point E on AC draw EF parallel to the plane XX, and cutting BD in F; then $\frac{AE}{EC} = \frac{BF}{BD}$ (**226**), and EF will be another element of the surface.

Draw EG perpendicular to AZ and FI perpendicular to WW, then GI will be parallel and equal to EF. We have also

$$\frac{AE}{EC} = \frac{AG}{GH}, \text{ and } \frac{BF}{BD} = \frac{BI}{IK};$$

therefore $\frac{AG}{GH} = \frac{BI}{IK}$, consequently HK and GI cut AB in the same point R.

Now draw any plane parallel to YY, cutting AB in L; its intersection with XX will be parallel to AZ, and will cut GI in N, HK in M, giving $\frac{LN}{NM} = \frac{AG}{GH} = \frac{GE}{HC}$. This plane also cuts the parallelogram GF in NO parallel and equal to GE, and the parallelogram HD in MP parallel and equal to HC. Therefore we have $\frac{LN}{NM} = \frac{NO}{MP}$; which proves that LOP is a right line, intersecting the three elements AB, EF, CD. Also, the elements CD, EF, AB, are proportionally divided by the planes parallel to YY, so that $\frac{CP}{PD} = \frac{EO}{OF} = \frac{AL}{LB}$.

If then AB and CD be taken as directrices, and YY as a plane directer, the resulting surface will be identical with that having AC and BD as directrices and XX for a plane directer. The rectilinear elements CD, EF, etc., which are parallel to XX, are called elements of the **first generation**; those of the other set, as AC, LP, etc., which are parallel to YY, are called elements of the **second generation**. And from the preceding it appears that every element of either generation intersects all those of the other.

228. Vertex and Axis. It is clear that a plane through R parallel to YY, in Fig. 192, will determine an element RT of the second generation, parallel to DK, which is perpendicular to WW. In a similar manner (**226**) an element of the first generation may be found which shall be parallel to any line drawn in the plane XX and perpendicular to WW. Thus in Fig. 193, having found RT as above, draw AF perpendicular to WW, pass through C a plane parallel to XX, cutting YY in DQ, and draw CE, perpendicular to DQ and consequently equal and parallel to AF. Then EF,

equal and parallel to AC, cuts BD in N, and NM parallel to AF is the required element.

The elements RT, MN, therefore, lie in a plane, which is perpendicular to WW, the intersection of the two plane directers, and cuts it at I. The point O in which these elements cut each other is called the **vertex** of the surface; and the line UU, drawn through O parallel to WW, is called the **axis**. The plane determined by RT and MN, again, is tangent to the surface at the vertex O (**142**).

Obviously, the angle NOR between these two elements is equal to the angle NIR between the two plane directers: if this angle is a right angle, the surface is called a **right**, or **rectangular**, hyperbolic paraboloid; if not, the surface is called **oblique**.

229. In the absence of a model, the pictorial representation in Fig. 194 may aid in forming a conception of this surface. NS in the horizontal plane, and MT in the vertical plane, are divided into the same number of equal parts, and the right lines joining the

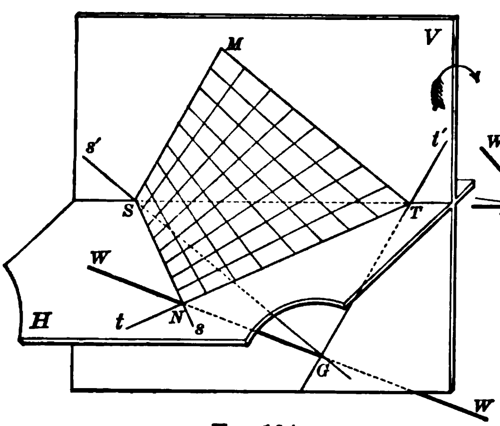


FIG. 194

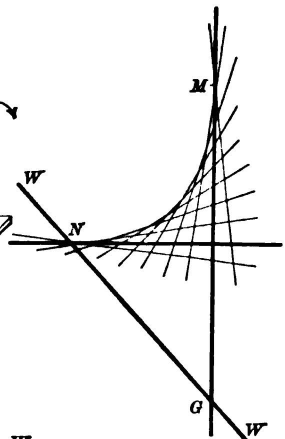


FIG. 195

corresponding points of division are elements of one generation; those of the other generation are determined by like treatment of SM in **V** and NT in **H**.

Draw Tt' parallel to MS, then tTt' is the plane directer of the first generation; in like manner Ss' parallel to MT determines sSs',

the plane directer of the second generation. The vertical traces of these two planes intersect at G on the vertical plane, their horizontal traces at N on the horizontal plane, thus determining WW, the intersection of the plane directers.

Since NS is an element of one generation, and NT one of the other, the horizontal plane is tangent to the surface at N (**142**); and for a like reason the vertical plane is tangent to it at M. This is most clearly shown in Fig. 195, which represents the same surface in projection on the profile plane.

230. The Principal Diametric Planes. Every hyperbolic paraboloid, whether rectangular or oblique, is symmetrically divided by two planes perpendicular to each other, passing through the axis, and bisecting the angles formed by the two elements which determine the tangent plane at the vertex.

Fig. 196 represents in profile the system of Fig. 193, for con-

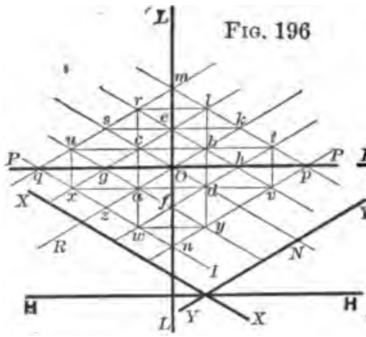


FIG. 196

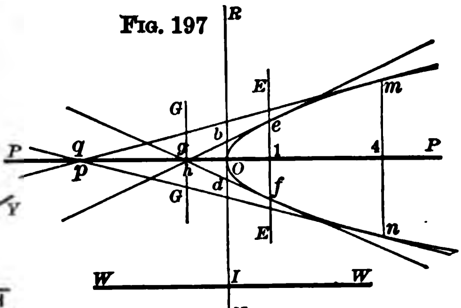


FIG. 197

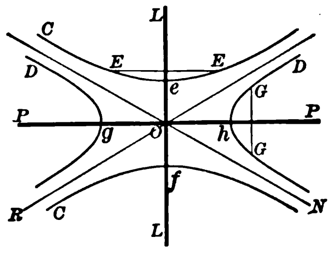


FIG. 199

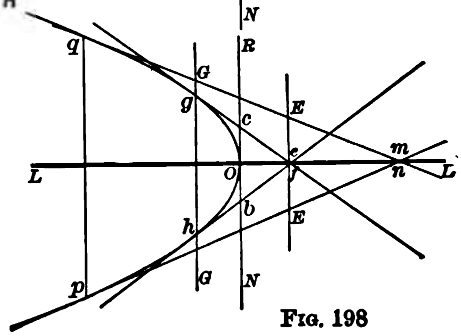


FIG. 198

venience so placed that the plane RN coincides with the paper, and that the plane directers are equally inclined to the horizontal plane HH, which contains their intersection WW, represented by the

point I: thus the axis is represented by the point O, and the elements of the two generations by the lines respectively parallel to XX and YY.

Set off on OR the points a, b, and on ON the points c, d, all equidistant from O; through a and b draw parallels to XX, and through c and d, parallels to YY: the intersections of these parallels determine the rhombus $egfh$. Through e and f draw the plane LL, and through g and h the plane PP, both perpendicular to the paper; these planes are perpendicular to each other, intersect in the axis O, and bisect the angles formed by OR and ON.

Now, the sides of the rhombus represent four elements of the surface; and since a, b, c, d, lie in the plane of the paper, the intersections e, f, g, h, do *not*, because those elements are inclined to that plane. Suppose e to lie at any distance behind it; then because eb, bh, hd, df, etc., are all equal, the point f must also lie at the same distance behind it; and the points g, h, at exactly the same distance in front of it.

Set off bk on RO equal to cs on NO, and complete the rhombus $mpnq$, then by the same reasoning it appears that m and n are at the same distance behind the plane of the paper, and p, q, at an equal distance in front of it. Produce the sides of the first rhombus to cut those of the other, then r, l, y, w, are at equal distances behind the plane, while u, t, v, x, are at equal distances in front of it. Then rl, sk, ut, etc., are chords of the surface parallel to the planes PP and RN; and the plane LL bisects them all. So also all chords parallel to LL and RN, as for example tv, ly, mn, etc., are bisected by PP. Consequently, as stated, the surface is symmetrical with respect to both planes; which are called the **principal diametric planes.**

231. The Gorge Lines. There may be an indefinite number of hyperbolic paraboloids having the same plane directers XX, YY, and the profile, Fig. 196, represents them all. Now, the point e may be at any distance beyond the plane of the paper; but if that distance be assigned, the inclination of the directrices to that plane is thereby fixed, and the surface fully determined.

Let OK be twice Ob; then since $el = ce$, l will be twice as far beyond the plane as e, and since $lm = lk$, m will be twice as far

beyond it as l, or four times as far as e; while the distance of m from the axis is but twice that of e.

Had Ok been three times Ob, m would have been three times as far as e from the axis, but nine times as far beyond the plane of the paper; and so on.

Consequently LL cuts the paraboloid in a curve, symmetrical with respect to the axis of the surface, having its vertex at O, extending to infinity beyond the plane RN, and having its abscissas proportional to the squares of the ordinates; that is to say, in a parabola. And in a similar manner it may be shown that PP cuts the surface in another parabola having its vertex at O, and extending in front of the plane RN.

232. Since the plane directers in Fig. 196 are equally inclined to the horizontal plane, PP is horizontal and LL is vertical. Fig. 197 is a projection upon the latter plane; in which the planes PP and RN are seen edgewise. Set off $O1$, the assumed distance of e from RN, and Og equal to it; also set off $O4$ and Op equal to four times $O1$. Taking the lengths of the chords ef, mn, from Fig. 196, draw the projections of the rectilinear elements through g, e; g, f; p, m; and p, n. Thus the parabola mOn is shown in its own plane; also, it is seen that ge, pm, are tangent to this curve at e and m, (the subtangents being bisected at the vertex): which is as it should be, because eg, eh, being elements of different generations, determine a plane tangent to the surface at their intersection e (**142**). The other parabola is shown in its true form in Fig. 198, which is a projection on the plane PP; the construction is obvious, the lengths of the chords gh, pq, being transferred from Fig. 196.

These parabolas are upon different scales, since the chords ef, gh, equidistant from the vertex, are respectively equal to the diagonals of a rhombus; this is because this particular surface is an oblique one: had it been rectangular, the rhombus would evidently have been a square, its diagonals equal, and the parabolas identical.

233. If in Fig. 197 a plane parallel to RN be drawn through any point as e, it will cut any pair of elements above e, as pm, qm, thus determining a chord of the curve of intersection, seen as EE in Fig. 198. By drawing other elements, other points of this

curve are easily found; and it is clear that e must be its vertex, because the elements ge, he, determine a plane tangent to the surface at that point, and that plane intersects the cutting plane in a line perpendicular to LL, which is the tangent to the curve at the point under consideration. The same plane will cut the lower part of the surface in a similar and opposite curve, as shown in Fig. 199, which is a projection on the profile plane RN. In this figure C, C, are the curves cut from the surface by the plane through e, and D, D, are those cut from it by a parallel plane through g, at the same distance from RN but on the opposite side. It can be proved that these curves are, in fact, hyperbolas, of which the centres lie in the axis of the surface, and the asymptotes are parallel to the elements OR, ON: but without discussing their mathematical peculiarities, it is evident from what precedes that they are convex toward the axis, and that their vertices lie upon the two parabolas shown in Figs. 197 and 198: which suffices to show that those parabolas are true **gorge lines**, as defined in (**129**).

234. The Plane a Limiting Form. Referring to Fig. 194, suppose the elements to be perfectly elastic lines, fixed at each end in the two planes **H** and **V**. Then if **V** be revolved about the ground line in the direction of the arrow, these elements will be lengthened, the curvature of the surface becoming less and less, until at the limit, when M falls in **H** beyond the ground line, the paraboloid will have been *extended* into a plane.

If on the other hand **V** be revolved in the opposite direction, the curvature of the surface will become greater and greater, the elements contracting, until M falls in **H** in front of the ground line, and the paraboloid will have *collapsed* into a plane.

235. The Warped Quadrilateral. Four right lines, connecting any four points not in the same plane, constitute what is sometimes called a *warped quadrilateral*, and **may** define a portion of a hyperbolic paraboloid: not necessarily, however, since there are other warped surfaces which are doubly ruled, and such a quadrilateral can evidently be drawn upon any one of them.

But supposing that they do; it is to be remarked that the same four points may be joined two and two by right lines in three different ways, thus forming three different quadrilaterals, as shown

in Figs. 200, 201, and 202, and determining as many diverse hyperbolic paraboloids.

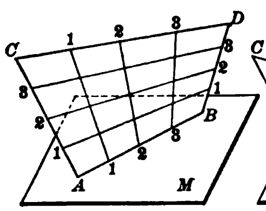


Fig. 200

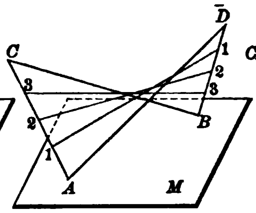


Fig. 201

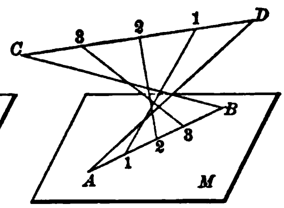


Fig. 202

236. Representation of the Surface. In Fig. 203, draw PD, MN, any two right lines of definite length not lying in the same plane, and divide each into the same number of equal parts; the lines joining the corresponding points of division will be elements of a hyperbolic paraboloid (**226**).

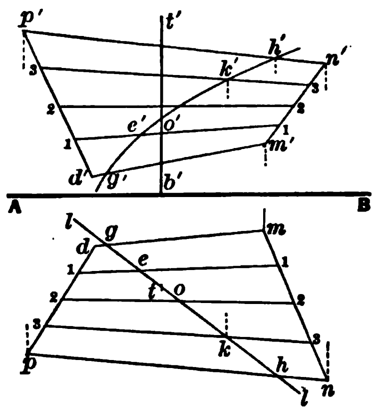


Fig. 203

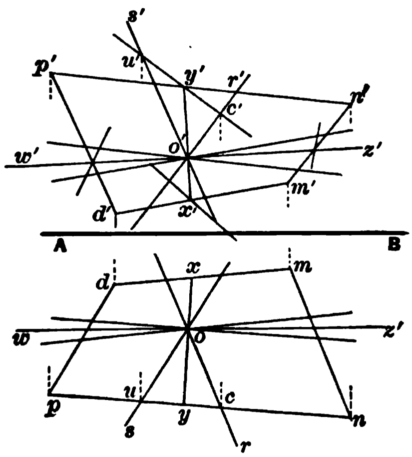


Fig. 204

To assume a point upon this surface. Assume the horizontal projection, as o, Fig. 203; the point must then lie upon a vertical line through o; of which $b't'$ is the vertical projection. Through this vertical line pass any plane, as ll; it cuts the elements at G, E, etc., thus determining a curve of intersection, whose vertical projection $g'k'h'$ cuts $b't'$ in o', the vertical projection of the assumed point: and in a similar manner the horizontal projection may be found if the vertical projection is assumed. In either case

there may be two determinations, since a right line may pierce the surface in two points; but no more.

The same process, obviously, is applicable to any other warped surface.

To draw an element through the point thus found. In Fig. 204, the quadrilateral PM and the point O, in order to avoid confusion in the diagram, are copied from Fig. 203. Through O draw OR parallel to MN, and OS parallel to PD; find as in Fig. 83 the points X and Y, in which the plane of these two lines cuts DM and PN: then XOY is an element of the surface.

To draw a plane tangent to the surface at this point. Draw through O a plane parallel to DM and PN, and find as above the points in which it cuts PD and MN; then WOZ drawn through those points is an element of the other generation, and the plane determined by XY and WZ is tangent to the surface at O (**142**): and its traces if required may be found in the usual manner.

237. Through any two given right lines not in the same plane, **any number** of hyperbolic paraboloids may be passed; for taking those two lines as directrices, there may be an infinite number of plane directers. This may be seen from another point of view, by considering that the ratio between the lengths of PD and MN, in Fig. 203, is entirely arbitrary; and that any change in this ratio must modify the form of the surface, since one of the plane directers is always parallel to DM and PN, and the other to PD and MN. It follows from this that if any two right lines not in the same plane be drawn of indefinite length, and equal spaces be set off upon each of them by any two scales of equal parts, whatever the ratio between the units of those scales: the right lines joining the successive points of division will be elements of a hyperbolic paraboloid.

238. Practical Applications. The hyperbolic paraboloid is sometimes met with, forming the basis if not the actual surface of a practical structure, without revealing its true nature to the casual observer. Thus in Fig. 205, **A** is a front view and **B** is a top view of the "pilot," or "cow-catcher," of an American locomotive, drawn of course in skeleton; this being divided by the central vertical plane LL, **C** is a view from one side, and **D** a view from the

other side, of that plane. Considering first the part on the left of LL in view **A**, and regarding the warped quadrilateral *aoeb* as lying upon a surface of this kind; then by subdividing *ao*, *be*, as above explained, we determine a series of elements parallel to LL, as

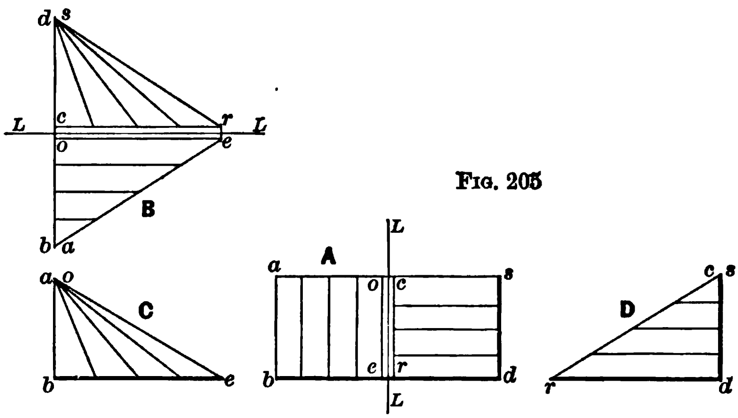


FIG. 205

shown in view **C** and in the lower half of view **B**. Upon the other side of LL there is a warped quadrilateral *scrd*, symmetrical to the first; in this case, by means of corresponding subdivisions of *cr* and *sd*, a series of horizontal elements is determined, as shown in view **D** and in the upper half of view **B**.

The pilot, then, is composed of two symmetrically placed hyperbolic paraboloids, of which the horizontal plane and the vertical plane LL are the common plane directers; and the bars may lie either in vertical planes or in horizontal planes: both arrangements have been used, the preference being given to the former.

239. This surface has also been employed in designing the bow of a boat, as shown in Fig. 206, where the water-lines are shown at **A**, while **B** is the sheer plan and **C** the body plan. The warped quadrilateral *dbco* being treated as in Fig. 200, the lines *xx*, etc., as well as the frames *nn*, etc., are right lines, as shown not only in **B** but in the left-hand part of **C** and the lower half of **A**. But the true *water-lines*, 1, 2, 3 (or horizontal sections, as shown in the right-hand half of **C**), are not straight, but what is technically called *hollow*,—that is, outwardly concave, as seen in the upper half of **A**. This circumstance is obviously due to the fact that *db* in this case is

not horizontal, but droops as it recedes from *do*, the vertical line of the stem; which is most clearly shown in the sheer plan **B**. And

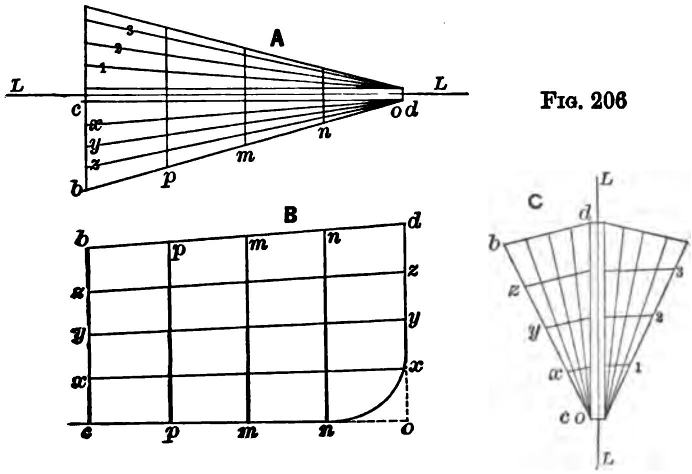


FIG. 206

it is equally clear that if *db* had been horizontal, all the water-lines would have been straight instead of concave.

THE CONOID.

240. The Conoid is a warped surface, with a plane directer, and two linear directrices, one of which is a right line, and the other a curve. Thus the elements of the surface, instead of converging to a single point as in the cone, pass through all the points of the rectilinear directrix. If that directrix is perpendicular to the plane directer, it is called the **Line of Striction**, and the surface is called a **Right Conoid**.

The definition being perfectly general, the curved directrix may be of single or of double curvature. But the term conoid is frequently used in a limited sense, as including only those surfaces in which this directrix is a closed curve, lying in a plane perpendicular to the plane directer.

241. In Fig. 207 is represented the most simple of the conoids, which bears to others the same relation that the cone of revolution has to all other cones; and in all probability gave the name to the class. This is a right conoid, of which the plane directer is **V**, and

the curved directrix a circle lying in **H**; the line of striction intersects at O the axis of this circle, which is evidently a line of symmetry, and may be called the axis of the surface. It is apparent that this surface is symmetrically divided by two planes through the axis, one of which is parallel, and the other perpendicular, to the plane directer; also that it is divided by the line of striction into

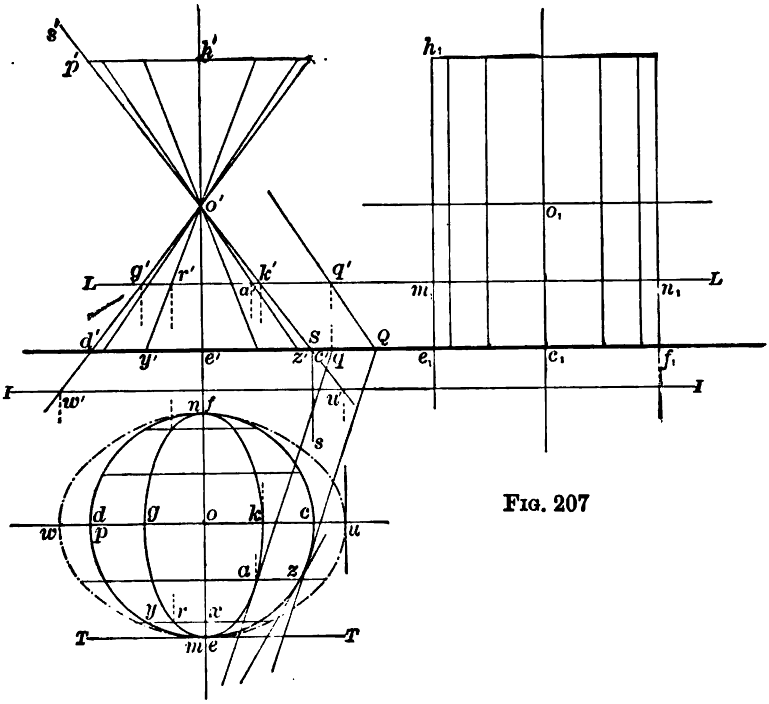


FIG. 207

two nappes, like a cone, which are similar and symmetrically placed.

The intersection of this conoid by a plane perpendicular to the axis is an ellipse.

Thus, let LL be such a plane, lying between the directrix and the line of striction: then it is seen that in the horizontal projection the ordinates go, rx, are less than, but directly proportional to, the corresponding ordinates do, yx, of the circle; therefore the curve is an ellipse whose major axis mn is equal to the diameter of the circle. If the elements be extended, and cut by a horizontal plane, as II, more remote than the directrix from the line of stric-

tion, the ordinates will be greater than those of the circle, and the *minor* axis of the elliptical section shown in a dotted line will be equal to the diameter of the directrix.

242. Planes Tangent all along Rectilinear Elements. It is very easy to draw the plane tangent at any point, as A for instance, since it must contain the element through that point, and also the tangent at the same point to the elliptical section by a horizontal plane; the horizontal trace of the required plane being drawn through z, the foot of the element, and parallel to that tangent, the vertical trace is drawn parallel to $a'z'$. And it is obvious that in general such a plane will not be tangent along the element, because the tangent to the circle at z is *not* parallel to the tangent to the ellipse at a.

But the plane TT, parallel to **V**, contains the vertical element EH, and also a tangent at one of the vertices to every elliptical section: it is therefore tangent all along the element.

So, again, the plane sSs', perpendicular to **V** and containing CP, is tangent to the surface all along that element; and two other planes possessing this peculiarity can be drawn on the opposite sides.

243. In Fig. 208, **V** is the plane directer, the curved directrix is the horizontal circle CC, and the rectilinear directrix DD is horizontal but inclined to **V**, and intersects at O the axis of the circle. The upper nappe of the surface is not represented, but the elements are continued below CC to the horizontal plane.

It is obvious on inspection that this surface is not symmetrical like the preceding one; the two nappes will not be similar, and the directrix is the only circular section.

In regard to the horizontal trace, we have, since the elements are proportionally divided by parallel planes, the values

$$\frac{fg}{gk}=\frac{sv}{vw}=\frac{oc}{cu}=\frac{an}{nz}=\frac{de}{em};$$

therefore that trace is an ellipse, and the same is true of a section by any horizontal plane as LL.

244. Since there must be a vertical element at R and another at S, it follows that rs will be a common diameter of all these

ellipses, as well as of the circle. Draw *gn*, a diameter of the circle perpendicular to *rs*, and through *g* and *n* draw the projections of two elements, determining the points *k* and *z* in the horizontal trace. Then regarding *rs* as a diameter of the ellipse, *kz* will be

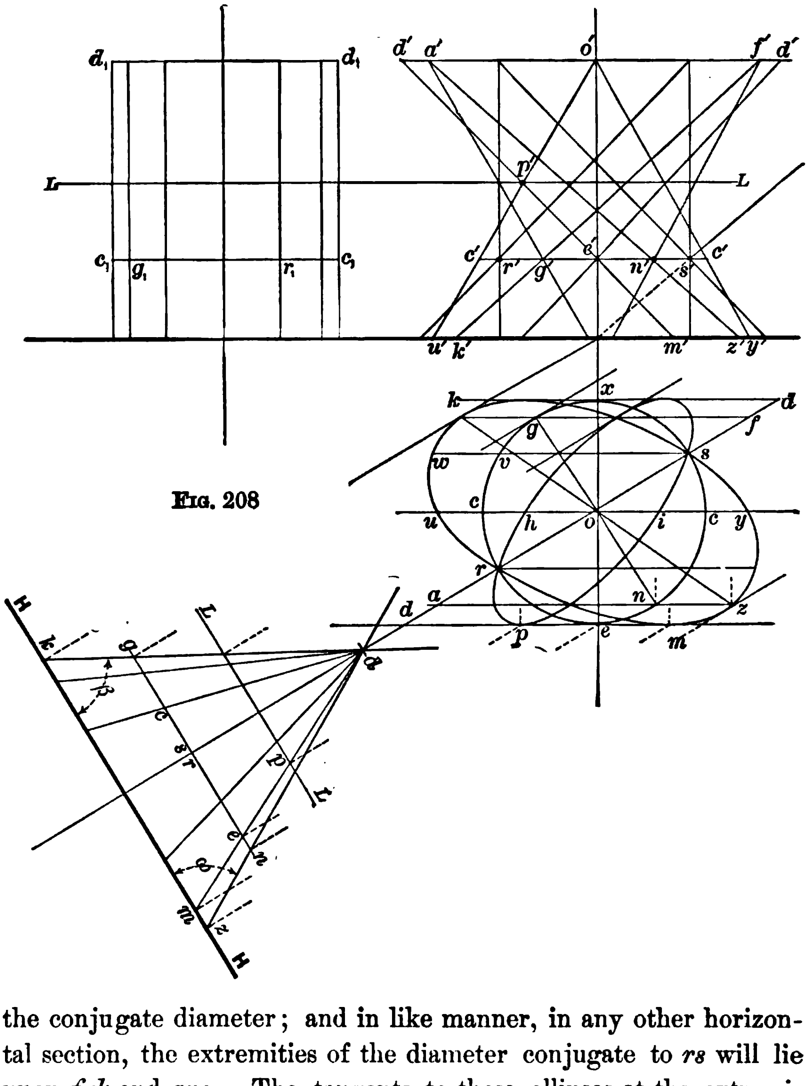


FIG. 208

the conjugate diameter; and in like manner, in any other horizontal section, the extremities of the diameter conjugate to *rs* will lie upon *fgk* and *anz*. The tangents to these ellipses at the extremities of these conjugate diameters are parallel to *rs*, consequently the planes determined by *DD* and the elements *FK*, *AZ*, are tan-

gent to the surface all along those elements; the true angles β, ω, which these planes make with **H**, are seen in the supplementary projection.

Draw cc, ex, two conjugate diameters of the directrix, respectively parallel and perpendicular to **V**. The elements through the extremities of the first determine the points u, y, of the horizontal trace; then reasoning as before, m, the extremity of the diameter conjugate to uy, must lie upon the element through e: so also does p, the extremity of a diameter conjugate to hi, in the section by the plane LL. From which it appears that, in this case also, two planes parallel to **V** can be drawn, each tangent all along an element.

245. Limiting Forms of the Conoid. These two examples illustrate sufficiently for our purpose the peculiarities of this surface, which, however, is susceptible of variations without number, since the straight directrix may have any position, and the curved one any form; in consequence of this, it happens that, in special cases, warped surfaces of other classes are also in fact conoids.

The less the inclination of the straight directrix to the plane directer, the more acute will be the angles between it and the elements; and at the limit, when that directrix is parallel to the plane directer, the elements will become parallel to it and to each other. In that event, if the other directrix be of double curvature, or lie in a plane which cuts the rectilinear directrix, the conoid will become a cylinder; if it lie in a plane parallel to the rectilinear directrix, the surface will degenerate into a plane.

THE HYPERBOLOID OF REVOLUTION.

246. This surface is generated by a right line, which revolves about an axis lying in a different plane. In Fig. 209, let the axis be vertical, the point o being its horizontal, and $w'o'y'$ its vertical projection; and let the generatrix MN be parallel to **V**: then each point of MN will describe a horizontal circle, of which the true radius is seen in the horizontal projection, and the exact altitude in the vertical projection.

Thus, the point N describes a circle whose radius is on; the vertical projection of this circle is determined by setting off on each

side of the axis, on the horizontal through n', the distance $y'k' = on$: similarly, the vertical projection of the circle described by the point E is $f'e'f'$, where $u'f' = oe$, and so on. The circle of least diameter is that described by the point C, since oc is the common perpendicular of the axis and the generatrix. This circle,

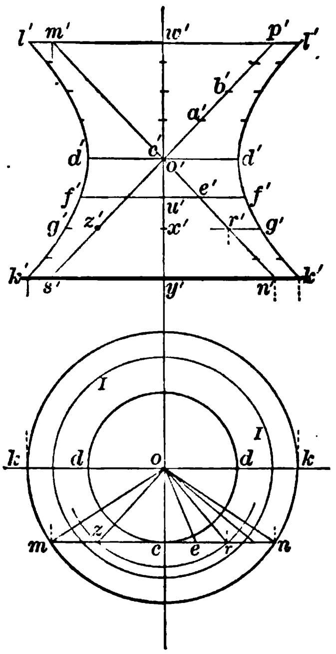


Fig. 209

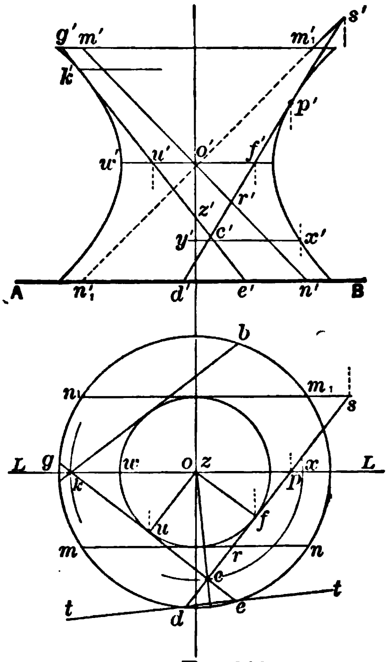


Fig. 210

whose vertical projection is $d'c'd'$, is called the **gorge circle**, and its plane is called the **gorge plane**. Since points of MN which are equidistant from C are also equally distant from the axis, and therefore describe equal circles, it follows that the portion $d'l'$ of the contour above $d'd'$ is precisely like the portion $d'k'$ below that line; and that the surface is symmetrically divided by the plane of the gorge.

247. Practical Suggestions. This surface has practical applications in mechanism; and it is proper here to point out most emphatically, that in constructing the contour it is not only useless, but worse than useless, actually to *draw* the vertical projections $d'd'$, $f'f'$, etc., thus covering the paper with superfluous lines to be subsequently erased: no good draughtsman will do it, nor will any

good instructor permit pupils to adopt such slovenly methods. It is not even necessary to draw the horizontal projections of the radii; having marked the projections of any point of *MN*, as r, r', mark on the axis the point x' on the horizontal through r', then taking the distance or in the compasses, a very short arc only is drawn as as g', g', on each side, with x' as a centre; the precise points on these arcs being finally marked on the horizontal through x'.

248. This Surface is Doubly Ruled. The portion here represented is limited by two planes equidistant from the gorge plane; which may be called its bases. Projecting m to the lower base at s', and n to the upper base at p', it is seen that mcn is the horizontal projection of a second line whose vertical projection is $s'c'p'$. This line makes the same angle as *MN* with the plane of the base, but slopes in the opposite direction; moreover, it is also a generatrix of the surface, because each of its points, in revolving around the axis, describes the same circle as a corresponding point of *MN*: for instance, Z and R, on the same side of the gorge and equally distant from it, are also equidistant from the axis, and describe the same horizontal circle. This facilitates the construction of the contour: thus a' is vertically over e', consequently those two points are equidistant from c', and the circles drawn through them will have the same radius oe; the radius of the circle through b', vertically over r', is equal to or, and so on.

249. Through any point of the surface, then, two elements can be drawn; and it is apparent that either of them if produced will intersect all those of the other generation except that one which is parallel to it, for the simple reason that they do lie upon the same surface; and it may be said to intersect that one at an infinite distance. From this it follows that if any three elements of either generation be taken as directrices, any element of the other may be taken as a generatrix, whose motion will produce the same surface. Considered in this light, the surface is one with three rectilinear directrices; regarded as a surface of revolution, it is one with a **cone director** and two circular directrices,—the former being a right circular cone whose angle at the vertex is $s'c'n'$ in Fig. 209; the base of which is *II* in the horizontal projection: when thus situated this cone is obviously asymptotic to the surface.

If a plane be drawn cutting this cone directer in an ellipse, a parabola, or an hyperbola, a plane parallel thereto will cut this hyperboloid in a curve of the same kind.

250. To assume a point on the surface. In Fig. 210, the contour having been found as above, assume the vertical projection c'; then a horizontal through c' determines the radius $y'x'$ of the circle upon whose circumference c must lie. Conversely, if the horizontal projection c be assumed; a circle through c, about centre o, will be cut by the meridian plane LL parallel to **V**, in the point x; which projected to the contour at x' will determine the altitude of the vertical projection c'.

Otherwise, if the horizontal projection c be assumed: Draw through c two tangents to the gorge circle; these are the horizontal projections of the two elements which pass through the assumed point. One of these pierces the lower base in D and the gorge plane in F; the other pierces the lower base in E and the gorge plane in U; and their vertical projections $d'f'$, $e'u'$, intersect in c', the vertical projection of the assumed point.

To draw a plane tangent to the surface at any point. Draw through the point an element of each generation. Thus in Fig. 209, the two elements whose horizontal projection is mn determine the plane tangent to the surface at the point C on the gorge circle. And in Fig. 210 the elements DF, EU, determine the plane tangent at the point C; in this figure the lower base coincides with **H**, consequently tt, drawn through d and e, is the horizontal trace of the plane. This trace is perpendicular to the radius oc, which fact enables us to draw it with precision, even when d and e are very close together; and indeed it enables us to dispense altogether with one of the elements through C, since, the direction being known, either one of the points, d or e, suffices to locate the trace.

251. It is to be observed that, having determined the circular paths of any two points in the generatrix, as for instance M and N in Fig. 209, the projections of any number of elements might have been drawn; and instead of constructing the contour line $k'd'l'$ by points as above, we could then have drawn it as the envelope of the vertical projections of those elements (**170**), each of which would be tangent to it at a point in the meridian plane parallel to **V**: thus in

Fig. 210, the element EG has but one point, K, in common with the meridian curve, and no point either of the element or the surface lies to the left of that curve; consequently k' is a point of tangency in the vertical projection.

252. The Meridian Curve is an Hyperbola. In Fig. 209, the intercepts $e'f'$, $r'g'$, etc., diminish as they recede from the gorge. In fact, each intercept, as $r'g'$ for instance, is the difference between the hypothenuse or and the base cr of a right-angled triangle whose altitude oc remains constant. Let the hypothenuse $= \mathbf{h}$, the base $= \mathbf{b}$, the altitude $= \mathbf{a}$; then $\mathbf{a}^2 = \mathbf{h}^2 - \mathbf{b}^2 = (\mathbf{h} + \mathbf{b})(\mathbf{h} - \mathbf{b})$, $\therefore \mathbf{h} - \mathbf{b} = \frac{\mathbf{a}^2}{\mathbf{h} + \mathbf{b}}$. Now the value of this fraction diminishes as the denominator increases, and reduces to zero when $\mathbf{h} + \mathbf{b}$ becomes infinite: consequently $c'n'$, $c'p'$, are *asymptotes* to the meridian curve $k'd'l'$, of which d' is the vertex and the horizontal line through c' is the axis.

In Fig. 210, MN, M_1N_1, are two positions of the same generatrix, both parallel to **V**, but on opposite sides of the surface; thus, $o'n'$, $o'm_1'$, are asymptotes to the contour. Draw any element DF of the other generation, cutting MN at R and M_1N_1 at S. This element pierces the meridian plane LL at P, and $d's'$ is tangent at p' to the contour (**251.**). Now, $rp = ps$, consequently $r'p' = p's'$; that is to say, the intercept between the asymptotes is bisected at the point of contact: which is a property peculiar to the hyperbola.

Whence it appears that the contour lines $k'd'l'$, $k'd'l'$, in Fig. 209, are the opposite branches of an hyperbola of which the centre is o' and the major axis is $d'd'$; and that the surface may be generated by revolving this hyperbola about its conjugate axis. This surface being unbroken, is distinguished as the hyperboloid of revolution *of one nappe;* the same hyperbola by revolving about the major axis will generate an hyperboloid of two separated nappes,—which however is obviously a double-curved surface.

225a. In Fig 210a, the planes TT, LL, parallel to V and to the axis, and tangent to the circle of the gorge, each cut from the surface two rectilinear elements, whose vertical projections coincide and are asymptotes to the meridian hyperbola. Draw two other

planes YY, WW, parallel to the others and equidistant from the axis but nearer to it; the sections will evidently be similar curves, the projections of those on the right-hand side of the axis coinciding in $e'v'h'$. Any rectilinear element of the surface, as RS, pierces these planes in the points O, C, M, N; and since $oc = mn$, it fol-

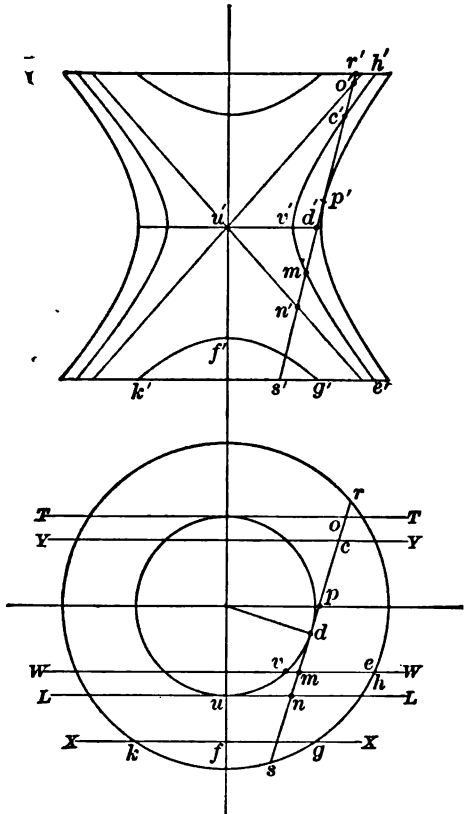


FIG. 210 a

lows that $o'c' = m'n'$; consequently $e'v'h'$ is also an hyperbola whose asymptotes are the same as those of the meridian outline.

A plane parallel to these but farther from the axis, as XX, will cut the surface in an hyperbola whose vertical projection $g'f'k'$ will have the same asymptotes, but its major axis will be vertical instead of horizontal.

253. In Fig. 209, the first position of the generatrix, MN,

was parallel to **V**; but in applications of this surface as an auxiliary, the given generatrix may not be so conveniently situated. To illustrate: suppose, in Fig. 210, the vertical axis and the line EG to be given, from which data the surface is to be constructed. Draw ou perpendicular to eg; this is the radius of the gorge circle, the altitude of whose plane is found by projecting u to u' on $e'g'$; this done, MN is drawn parallel to **V**, whence the asymptotes are determined and the contour constructed as in Fig. 209.

If we regard $e'g'$ in Fig. 210 as the vertical trace of a plane perpendicular to **V**, and tangent to the hyperboloid at K; then that plane must contain the companion generatrix passing through K, whose horizontal projection is kb; also, it will cut the meridian plane LL in a right line which intersects the axis of the surface at Z, and it is apparent that KZ is the true tangent at K to the hyperbolic outline. These considerations will be of use in employing this surface as an aid to the solution of the following problem.

254. PROBLEM. *To pass through a given right line a plane tangent to any given surface of revolution.*

Analysis. Revolve the given line about the axis of the given surface, thus generating a warped hyperboloid of revolution; the required plane will be tangent to this hyperboloid at some point of the given line (**142**). It will also be perpendicular to the meridian plane through the point of contact, on either surface; but the two surfaces having a common axis, these meridian planes will coincide, and will cut from the common tangent plane a right line, tangent to both meridian curves. This tangent and the given line determine the required plane.

Construction. In Fig. 211, let CN be the given line, through which it is required to pass a plane, tangent to the surface of revotion with a vertical axis, whose contour is shown in the vertical projection. Find the radius and altitude of the gorge circle, and construct the hyperboloid, as explained in (**253**). Draw, mechanically, $s's'$ tangent to the given contour and also to the hyperbola $k'g'l'$; this is the vertical trace of the required plane when revolved about the axis of the surface until it is perpendicular to **V**. The plane in this position contains an element C_1D_1 (the revolved position of the given line), which pierces the meridian plane LL par-

allel to V, at O_1 the revolved position of the point of contact with the hyperboloid. The revolved position of the point of contact

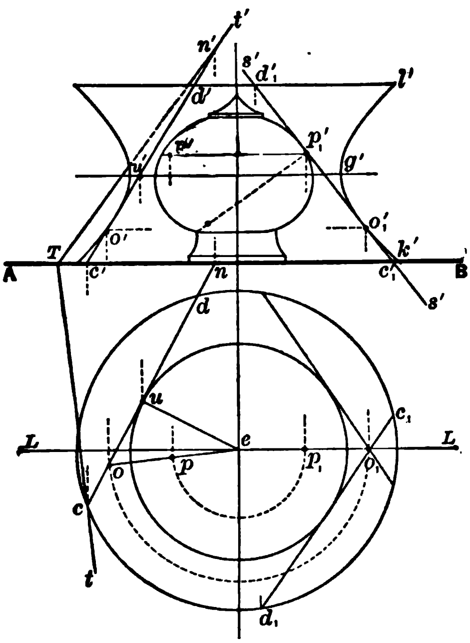


FIG. 211

with the given surface is vertically projected at p_1', found as in (**147**), and horizontally at p_1 on LL.

255. In the counter-revolution, O_1 returns in a horizontal circle to O on CD; o is at once located by setting off $do = d_1o_1$; then, drawing the radius oe, the horizontal trace of the required plane is tcT perpendicular to oe, and the vertical trace is Tt', passing through n', the vertical trace of the given line. The point of contact P may be located in a similar manner by setting off on oe the distance $ep = ep_1$; the vertical projection p' lies upon a horizontal line through p_1'.

Note. The common tangent line PO, mentioned in the argument, is not here made use of, since the directions of the traces can be determined without it, and in this instance more conveniently; but in other cases the use of that line might be essential.

256. Limiting Forms of this Surface. If in Fig. 209 the radius oc be gradually diminished, other things remaining unchanged, the

gorge circle will become less and less, the surface assuming a closer resemblance to a cone, which is the limiting form reached when the common perpendicular becomes zero and the generatrix intersects the axis.

If, oc remaining unchanged, we suppose MN to make a greater angle with the horizontal plane, the curvature of the surface will diminish, and it will more and more resemble a cylinder, which is the limiting form reached when the generatrix is parallel to the axis: the diameters of the gorge and the base circles then becoming equal.

If on the other hand the generatrix becomes horizontal, it must always lie in a plane perpendicular to the axis, which therefore is a third limiting form of this hyperboloid. The plane generated in this manner, however, cannot abstractly be considered to exist within the circumference described by the point C; which is in fact a true edge of regression.

THE ELLIPTICAL HYPERBOLOID.

257. Let the smaller circle E, in Fig. 212, be the horizontal projection of the gorge, the larger one, F, that of the upper and lower bases equidistant from it, and mn that of the generatrix, of an hyperboloid of revolution; whose vertical projection is constructed as above explained. If the ordinates of both these circles which are perpendicular to LL be reduced in the same proportion, the circles will be transformed into two ellipses, I and J; which are similar to each other, since the ratio of the major to the minor axis is the same in both. Now, if these ellipses be substituted for the original circular gorge and bases, a right line moving in contact with them all will generate a new surface, called an **elliptical hyperboloid.**

If we construct as in Fig. 209 the cone directer of the original hyperboloid of revolution, and treat its base in like manner, that cone will be transformed into the **elliptical cone directer** of the new surface. The sections of this cone and of the hyperboloid itself by parallel planes will be either ellipses, parabolas, or hyperbolas, according to the positions of the cutting planes.

258. The chord *ke* of the ellipse *J*, tangent to the ellipse *I* at the extremity of the minor axis, is the horizontal projection of an

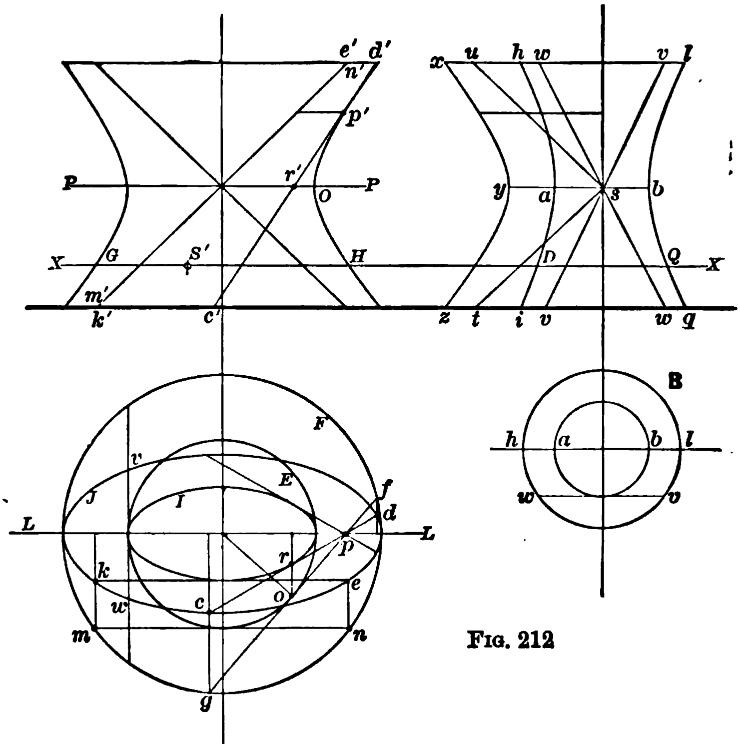


FIG. 212

element of this surface: this element is by construction parallel and equal to the element *MN* of the hyperboloid of revolution; and will have the same vertical projection, since *mk*, *ne*, are perpendicular to **V**.

Let *gf* be the horizontal projection of any other element of the hyperboloid of revolution; it is tangent at *o* to the circle *E*, and cuts *LL* in *p*. From *g* and *f* draw ordinates perpendicular to *LL*, cutting the ellipse *J* in *c* and *d*; and from *o* draw another, cutting the ellipse *I* in *r*: then since by construction *gc*, *fd*, *or*, are equal fractions of these ordinates, the chord *cd* will pass through *r* and *p*, and by a known property of the curve it will be tangent at *r* to the smaller ellipse.

Consequently *cd* is the horizontal projection of an element of

the elliptical hyperboloid, just as gf is of an element of the circular one; and since gc, fd, are perpendicular to **V**, these two elements will have the same vertical projection $c'd'$: which is also the vertical projection of another element, whose horizontal projection is drawn through p, tangent to the gorge ellipse on the opposite side.

From the preceding it follows that the elliptical hyperboloid is doubly ruled, and that when placed, as here shown, with the major axes of its bases parallel to **V**, its vertical projection will be identical with that of the circular one from which it is derived in the manner above set forth.

Evidently, the ordinates of the circles might have been increased, instead of being diminished, in any desired proportion, without affecting the argument: in which case the minor instead of the major axis would have been parallel to **V**.

In the profile, it is evident that hl, iq, will be equal to the minor axis of the larger ellipse, and ab to that of the smaller one. The contour in this view may logically be determined thus: Draw xyz the contour of the original circular hyperboloid; then ordinates from this contour perpendicular to the axis are to be reduced in the same proportion that was adopted in constructing the ellipses I and J. Treating in the same way the asymptotes st, su, to the curve xyz, we shall have two right lines, ww, vv; which are the two elements represented by wv in the horizontal projection. From this construction it is seen that sw, sv, are asymptotes to the contour hai; which can be shown to be an hyperbola by reasoning similar to that of (**252**).

260. To assume a point upon this surface. If the horizontal projection be assumed, the horizontal projection of an element through the point will be a tangent to the gorge ellipse; and the vertical projection of the point must lie upon the vertical projection of this element: it can therefore be determined directly.

But if the vertical projection be assumed, as at S'; it is then necessary to draw the section of the surface by a plane XX passing through the point and perpendicular to the axis. It follows from what precedes that this section will be an ellipse of which the major axis is GH in the vertical projection, and the minor axis is DQ in the profile. This ellipse will be seen in its true form in the hori-

zontal projection, and the horizontal projection of the assumed point must lie upon the curve.

To draw a plane tangent to this surface at a given point. The surface being doubly ruled, the tangent plane is determined by drawing through the point an element of each generation.

261. If, in Fig. 212, the ordinates perpendicular to LL in the horizontal projection had been *increased* instead of diminished in any given ratio, the circles would have been transformed into ellipses with their *minor* axes parallel to **V**. But, the preceding argument still holding true, the vertical projection of the resulting elliptical hyperboloid would have been identical with that of the original circular one: accordingly, the profile in the figure may be regarded as the vertical projection of an hyperboloid of revolution, in which the diameters of the gorge and the bases are respectively equal to the minor axes of the ellipses I and J; as indicated in the diagram **B**, drawn below the profile.

It follows, then, not only that from a given circular hyperboloid an infinite number of elliptical ones may be derived; but that the former is merely a special case, in which the axes of the three elliptical directrices become equal.

262. Limiting Forms of the Elliptical Hyperboloid. By reasoning analogous to that of (**256**), it will be apparent that when the generatrix intersects the axis, the surface will become an elliptical cone; when it is parallel to the axis, the surface will become an elliptical cylinder; and if it is tangent to the gorge ellipse, the surface will become a plane with an elliptical perforation.

263. The Hyperbolic Paraboloid a Special Case. A limiting form may be reached, however, in a different manner. In Fig. 213, let c be the axis and mrn the generatrix of an hyperboloid of revolution; the latter retaining its present position, suppose the axis to recede, as indicated by the successive positions c_1, c_2: it is clear that the difference between the radii cr and cm will become less and less, until when the axis is infinitely remote, those radii will be equal, the two circumferences will become one right line, and the surface will become a plane; which therefore is a limiting form in this case.

In Fig. 214, the gorge and base ellipses having been derived

from the circles shown, as in Fig. 212, let o and e be their corresponding foci: then from the mode of derivation it follows that the semi-parameters os, er, have the same ratio as the radii ca, cb. Now, the points a, b, o, s, remaining fixed, suppose again the axis to recede to c_1, c_2, etc. The curves, as has been seen, will always

FIG. 213

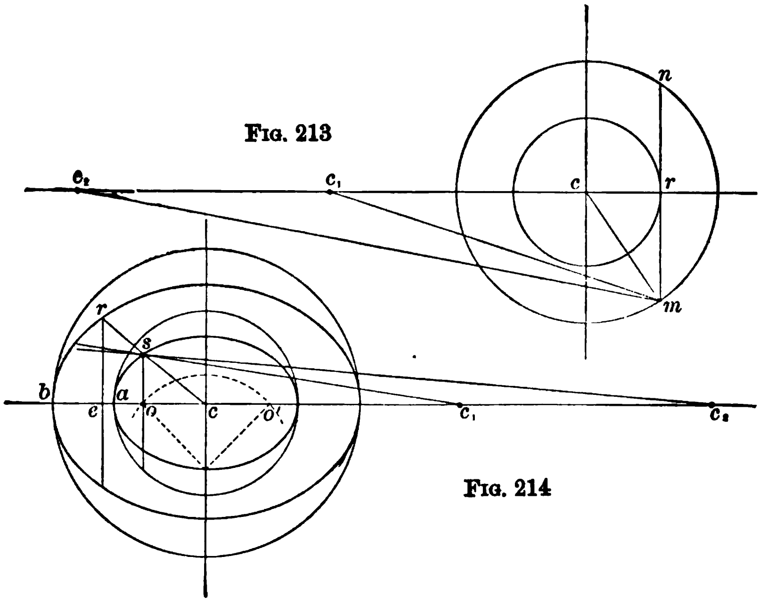


FIG. 214

be similar ellipses; and their semi-parameters approach equality as the axis recedes, becoming equal when its distance is infinite: but at this limit, the conjugate focus o' being also infinitely remote, the gorge ellipse becomes a parabola; and from the preceding argument it follows that the outer ellipse will also become a parabola *having the same parameter*. In these circumstances, the elliptical hyperboloid becomes a hyperbolic paraboloid; which in view of this mode of derivation might more consistently be called a **parabolic hyperboloid.**

264. It will now be clear that the gorge plane PP, and the plane LL, in Fig. 212, correspond in position to the principal diametric planes in Figs. 196–199; and supposing the axis of the surface to recede to the left, the gorge ellipse I will at the limit become the gorge parabola of Fig. 198, and the hyperboloidal con-

tour HOp' will become the other gorge parabola, shown in Fig. 197; O, the common vertex of the ellipse and the hyperbola in Fig. 212, will remain the common vertex of the two gorge parabolas: it will also be the **vertex** of the resulting hyperbolic paraboloid,—whose **axis**, however, has no relation to the axis of the original surface, since it is merely the common axis of the two gorge lines, being the intersection of the planes PP, LL.

265. The ellipse J, in Fig. 212, represents the intersection of the elliptical hyperboloid by either of two planes, parallel to PP and equidistant from it. And we observe that any chord cd of this ellipse which is tangent to the smaller one I, is bisected at r, the point of contact; which from the mode of derivation must hold true whatever the relative magnitudes of the corresponding axes,—and these again are independent of the distances from the gorge plane.

At the limit both ellipses become parabolas; whence the following problem may arise. Given, in Fig. 215, the parabola

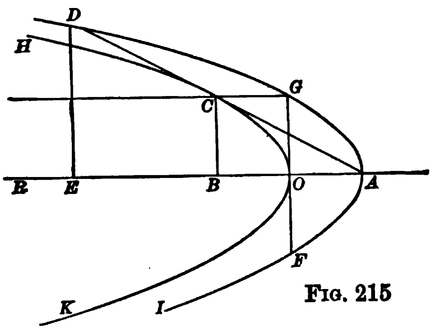


FIG. 215

HOK, of which O is the vertex and OR the axis: Required, to construct another parabola with the same axis, and a given vertex A, any chord of which, tangent to the given one, shall be bisected at the point of contact.

Any parabola having the same axis will satisfy the condition for the chord tangent at O; it remains then to find one which will satisfy it in regard to some other chord. Set off $OB = OA$, erect the ordinate BC, draw AC tangent to HOK, produce it, and make $CD = CA$; then D is a point in the required curve DAI. Draw

the ordinates DE, OG, then by construction $AE = 2AB = 4AO$, and since the curve is a parabola, $DE = 2GO$; but by construction $DE = 2BC$, consequently $GO = BC$. That is to say, the two parabolas are precisely alike; which agrees with the conclusion otherwise reached in (**263**).

266. In Fig. 216, the two parabolas *hok*, *dai*, in the horizontal projection, are the same as those of the preceding diagram; the

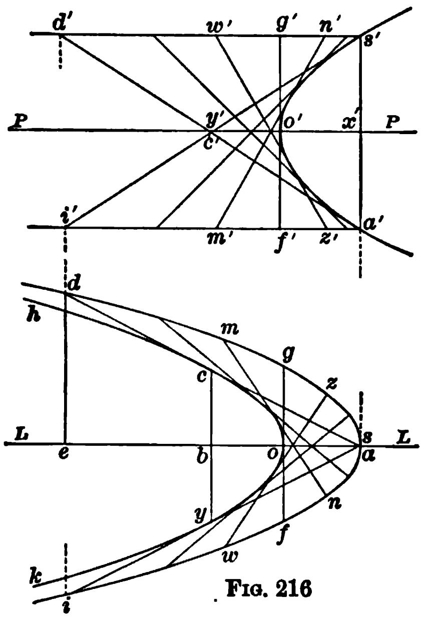


FIG. 216

vertical projection of the former is in the line PP, while $s'd'$, $a'i'$, parallel to PP and equally distant from it, are the vertical projections of the latter. Now, whatever the actual magnitudes of $x's'$ and $x'a'$, it is to be observed that $s'i'$, $a'd'$, will intersect at y' on PP, giving $y'o' = o'x'$, and will therefore be tangent to a parabola of which o' is the vertex; and in like manner it may be shown that $m'n'$, $w'z'$, and in short the vertical projections of all tangents to *hok*, will be tangent to the same parabola $a'o's'$; which lies in the vertical plane LL: the surface thus determined is, then, a hyperbolic paraboloid.

By similar reasoning it may be shown that sections by planes parallel to LL, will be parabolas precisely like $a'o's'$; that is to say, all sections of this surface by planes parallel to either of the principal diametric planes, will be similar and equal to the gorge parabola which lies in that plane.

267. If we suppose the parameter of the gorge hok to be indefinitely reduced, other things remaining unchanged, that parabola will ultimately become a right line, and the surface will become a plane coinciding with LL, its elements being then tangents to $a'o's'$. Similarly, if the horizontal gorge remain unchanged, while the parameter of $a'o's'$ is reduced, the elements will ultimately become tangent to hok, and the surface a plane coinciding with PP.

268. This derivation of the hyperbolic paraboloid, though showing it to be doubly ruled, gives thus far no evidence of the existence of plane directers. But if it be established that the elements of each generation divide those of the other proportionally, it must follow that those of either set lie in a series of parallel planes.

If this relation is true as to the elements in space, it must also be true of their projections, in which all parts are equally foreshortened; in other words the tangents to the gorge parabolas in Fig. 216 must divide each other proportionally. That they do so, is a fact made use of in the familiar construction shown in Fig. 217; the sides EC, CD, of the isosceles triangle ECD being divided into the same number of equal parts, the points of division are numbered in opposite directions, and the corresponding numbers joined by right lines: the envelope of these lines is the parabola EAD. The following demonstration of this depends upon two properties of the parabola, viz., that the abscissas are proportional to the squares of the ordinates, and that the subtangent is bisected at the vertex.

269. In Fig. 218, let A be the middle point of BC, which bisects the vertical angle of the isosceles triangle EDC. On the equal sides set off $DF = CH$ of any length at pleasure, draw FH cutting BC in G, and on BC set off $AM = AG$. Perpendicular

to BC draw FO, ML cutting FH in L, AP cutting FH in K and DC in P, and HI cutting BC in N and CD in I.

Now since BC is bisected at A, CD is tangent at D to a parabola of which A is the vertex; and since MG is bisected at A, FH is tangent at L to a parabola of which A is the vertex. Also these parabolas have the common axis BC; but in order to prove

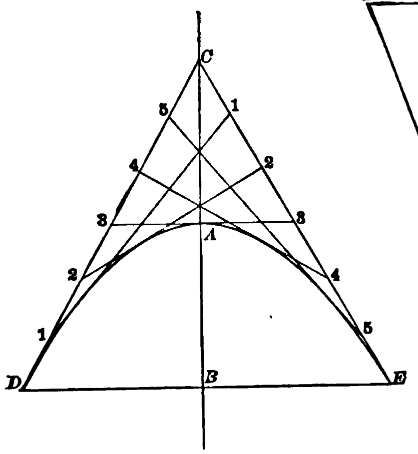


Fig. 217

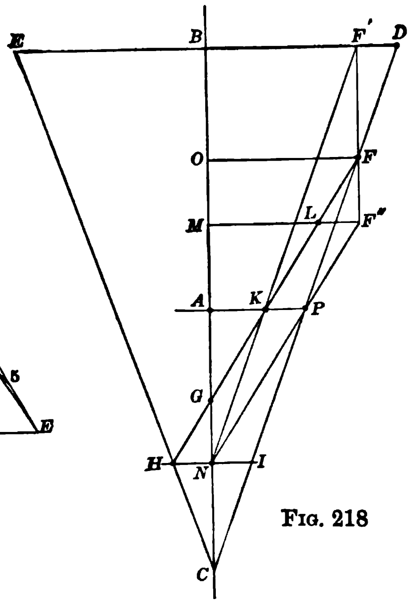


Fig. 218

that they are one and the same, it is necessary to show that $AM : AB :: ML^2 : BD^2$.

But $AM = AG$, $AB = AC$, and $ML : BD :: AK : AP$; consequently the above proportion may be written $AG : AC :: AK^2 : AP^2$; or, in fractional form, $\frac{AG}{AC} = \frac{AK^2}{AP^2}$.

In order to demonstrate this, we proceed as follows: Draw through F a parallel to BC, cutting BD in F', and ML produced, in F''; draw also NF', NF''. Then since $FF' = CN$, NF' is parallel to CD; and since $FF'' = NG$, NF'' is parallel to FH. Also, since FH, FI, are bisected by AP, we have $HN = NI = KP, = F'D, = LF''$; therefore NF' passes through K, and NF'' passes through P.

$$\left.\begin{array}{ll}\text{Then from similar triangles } GAK,\ NAP, \text{ we have} & \dfrac{AG}{AN}=\dfrac{AK}{AP}, \\ \text{and} \quad `` \quad `` \quad `` \quad NAK,\ CAP, \quad `` \quad `` & \dfrac{AN}{AC}=\dfrac{AK}{AP}.\end{array}\right\}$$

$$\therefore \frac{AG}{AC}=\frac{AK^2}{AP^2}. \qquad \text{Q.E.D.}$$

269a. These hyperboloids, viz., the circular, elliptic, and parabolic, are the only doubly-ruled warped surfaces, and also the only ones having three rectilinear directrices. Each has two gorge lines, lying in planes perpendicular to each other, and having a common vertex; and from the preceding it is apparent that if sections be made by planes parallel to and equidistant from either gorge, their projections on the plane of that gorge must coincide, be similar to the gorge curve and similarly placed, and satisfy the condition that any chord of the outer curve, tangent to the inner, shall be bisected at the point of contact. It has been shown that this condition can be satisfied by the conic sections; and it is not known that it can be satisfied by any other curves.

THE HELICOID.

270. We shall use the term **helix** in the restricted sense in which it is commonly employed; as designating the path of a point which, while revolving uniformly around an axis, also moves uniformly in a direction parallel thereto.

This curve, then, lies upon the surface of a cylinder of revolution, cuts all its rectilinear elements at the same angle, and becomes a right line when the cylinder is developed into a plane.

This being understood, we shall use the term **Helicoid** to designate any surface generated by a right line whose motion is determined by helical directrices lying upon concentric cylinders.

The right line, thus controlled, must necessarily have a motion of revolution; and it may either intersect the axis, like the generatrix of a circular cone, or remain always at a fixed distance from it, like that of an hyperboloid of revolution.

271. Helicoids of Uniform Pitch and of Varying Pitch. If all the helical directrices have the same pitch, every point of the generatrix will travel, in a direction parallel to the axis, at the same

rate; and that line will either lie in, or make a constant angle with, a plane perpendicular to the axis: the surface is then said to be of **uniform pitch.**

But the directrices may be of different pitches; in which case, the rate of **axial advance** being different for the various points of the generatrix, that line will continually change its inclination to the **transverse plane**; and the surface is then called a helicoid of **varying pitch.**

272. Right and **Oblique** Helicoids. The helicoids of uniform pitch may be subdivided into two classes. In Figs. 219 and 220, let the generatrix DE revolve uniformly around the vertical axis,

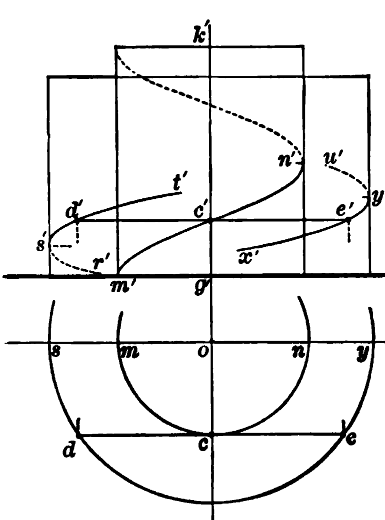


FIG. 219

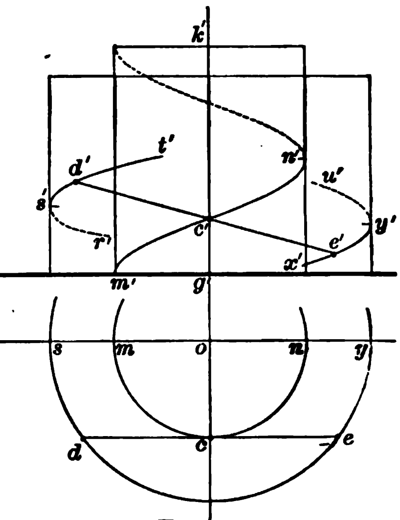


FIG. 220

while at the same time all its points move uniformly in a direction parallel to the axis; then the point C will describe the helix MCN lying on the inner cylinder, and the points D and E will describe helices of the same pitch lying on the outer cylinder; of which $r's't'$, $x'y'u'$, are the vertical projections. In both cases, the generatrix remaining at the fixed distance oc from the axis, will always lie in a plane tangent to the inner cylinder, touch that cylinder in a single point, and cut the vertical element through that point at a constant angle.

In Fig. 219, the generatrix is *perpendicular* to that element,

and therefore parallel to the transverse plane; that plane, then, is the plane directer of the surface, which is called a **right helicoid.** The radius oc being arbitrary, may be reduced to zero; in which case the generatrix intersects the axis, and the helicoid becomes a special form of the right conoid, the axis being the line of striction: this is a most familiar modification in practice, since it is the surface of the common square-threaded screw.

In Fig. 220, the generatrix is *inclined* to the vertical element of the inner cylinder through the point of contact. The surface is then called an **oblique helicoid**, and has a cone directer instead of a plane directer: if oc be reduced to zero in this case, the surface is that of the ordinary triangular or V-threaded screw.

273. Representation of the Helicoid. Since each of the points D, C, E, in the two preceding figures, describes a helix of the same pitch, it is easy to draw as many elements as may be desired. But the mere projection of a number of elements of indefinite length does not ordinarily convey an adequate idea of the form of any surface. This is better done by representing a limited portion, as in Fig. 221: the conditions here are similar to those in Fig. 220, and the surface is at once recognized as resembling that of the groove in an auger or a twist drill. This figure illustrates the principle already mentioned, that the visible contour is the envelope of all lines of a surface; it is, accordingly, tangent to the helical paths of the points M, N, etc., which are used for this determination in preference to the rectilinear elements: the latter would of course have served the same purpose, but practically would have been more confusing and very little if any easier to construct.

274. To assume a point on the surface. Supposing the axis to be vertical, as shown; then if the horizontal projection be assumed, an element can at once be drawn through the point. If the vertical projection be assumed, the point must lie on a perpendicular to **V** through the point; any plane through this line will cut the elements in points determining a curve, whose horizontal projection will cut that of the line, in the required horizontal projection of the point.

275. Peculiarities of Meridian Sections. If in Fig. 221 the surface be cut by a plane through the axis, perpendicular to **V**, the

section will be the curve $g_1c_1h_1$, shown in the profile. This curve is easily constructed by drawing elements of the surface, and finding the points in which they pierce the cutting plane: for example, the element XY pierces that plane at P; the altitude of p_1 is the same as that of p', and its distance p_1o_1 from the axis is equal to po.

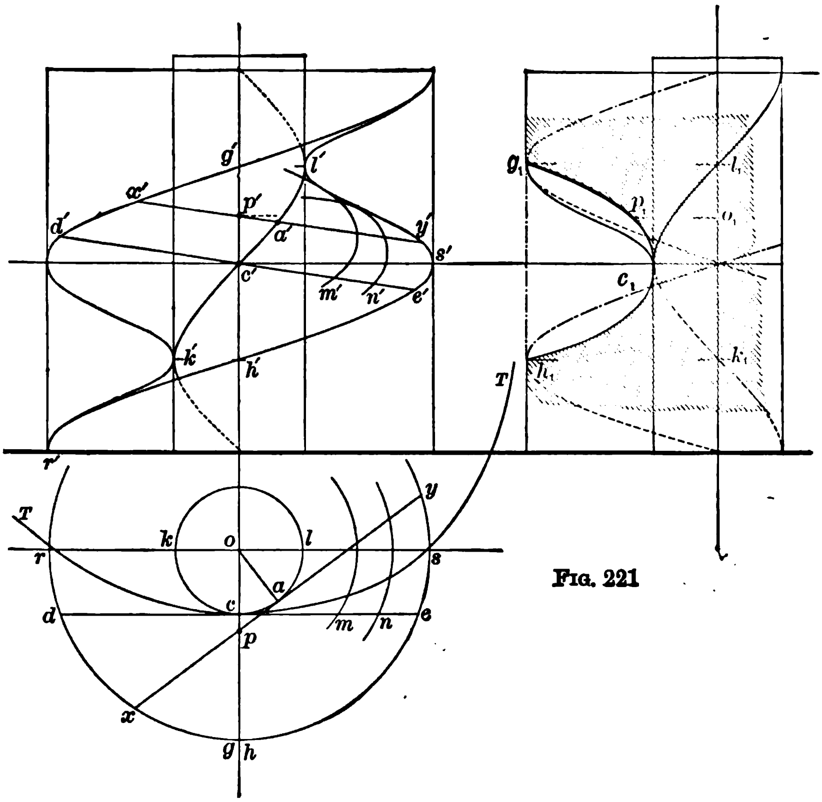


FIG. 221

This curve is convex toward the axis, and tangent at its vertex c_1 to the vertical element of the inner cylinder. This helicoid, then, surrounds a cylindrical well, within which it cannot extend; and since all the meridian sections are obviously alike, the surface is tangent to the cylinder all along the helix KCL, which is a true gorge line.

In Fig. 222, the generatrix is perpendicular to the element of the inner cylinder through the point of contact, as in Fig. 219; the diameters of the two cylinders, as well as the helical pitch, are

the same as in Fig. 221. This change in the position of the generatrix has caused the helices described by D and E to approach

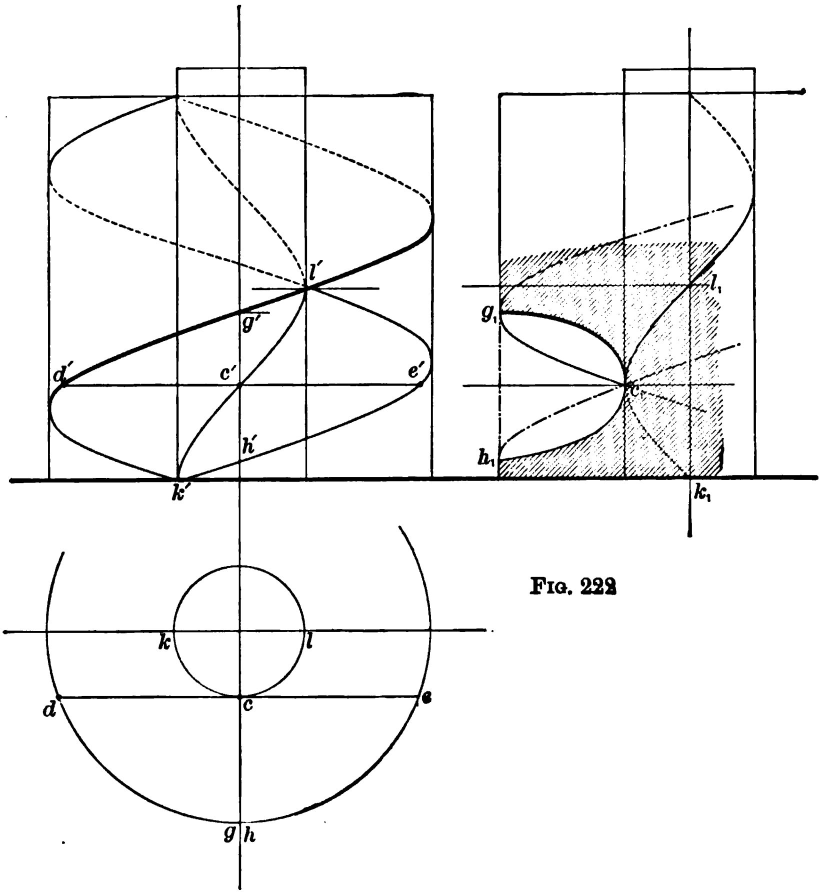


FIG. 222

each other, so that the breadth $g'h'$ of the groove is less than before.

Still the form of the meridian section is not greatly changed; and it is evident upon consideration that if the generatrix be lengthened, thus enlarging the outer cylinder, the curve $g_1c_1h_1$ will continue to expand as it recedes from the vertex c_1, though less and less rapidly; the curve having two horizontal asymptotes through l_1, k_1.

276. In Fig. 223, DCE is again inclined, but in the opposite

direction; and to such an extent that D now lies upon the lower and E upon the higher of the two helices on the outer cylinder.

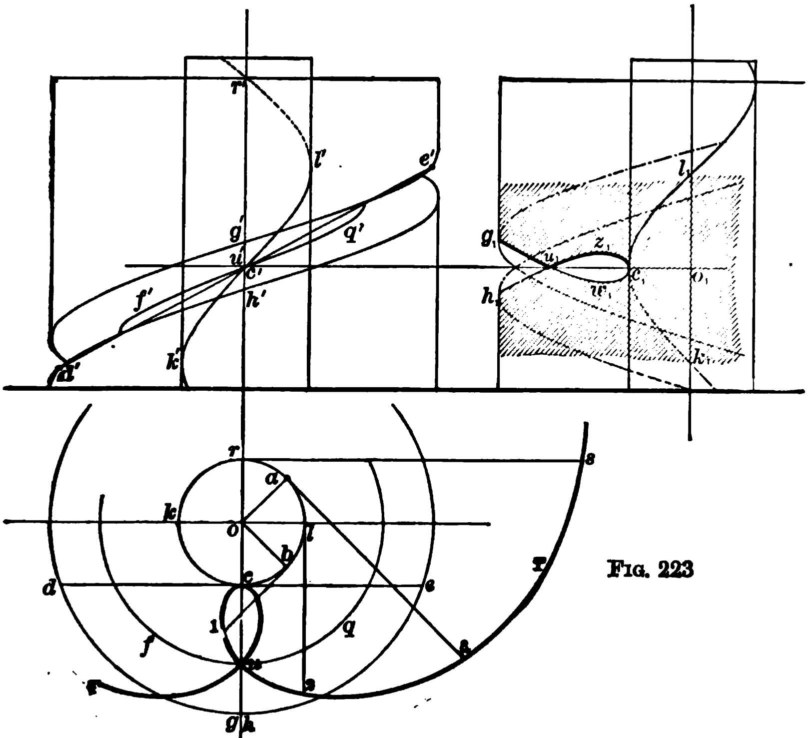


Fig. 223

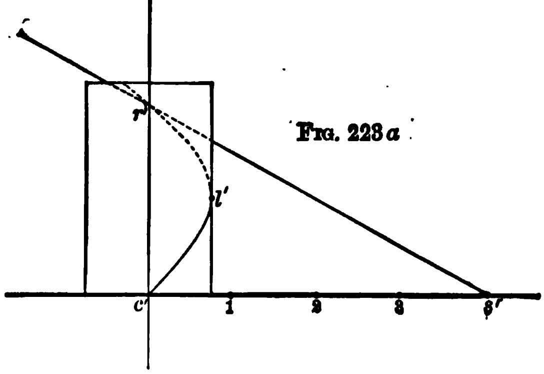


Fig. 223 a

The result is that the meridian section is a looped curve, crossing itself at a point u_1 between the two cylinders. Thus this helicoid

encloses not only the cylindrical well, to which it is tangent along

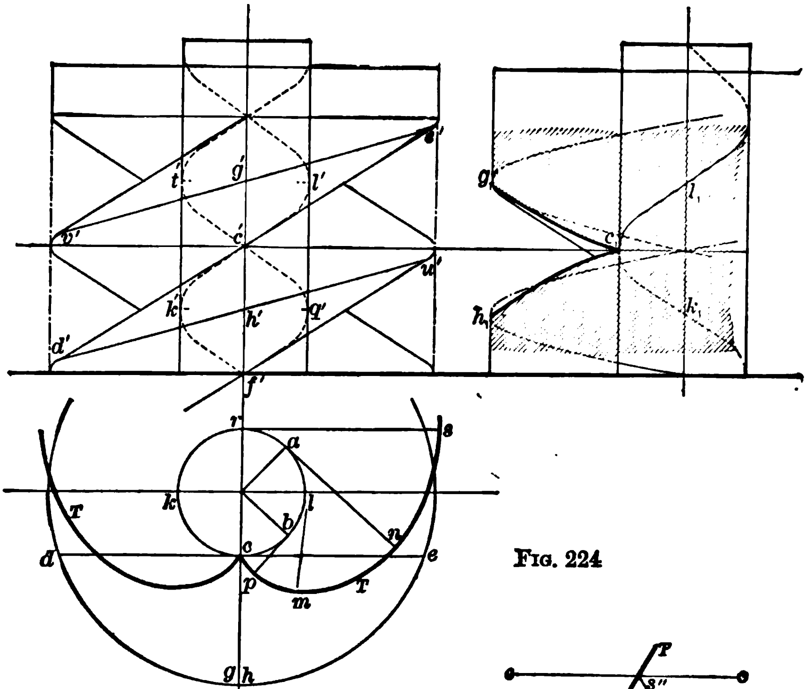


Fig. 224

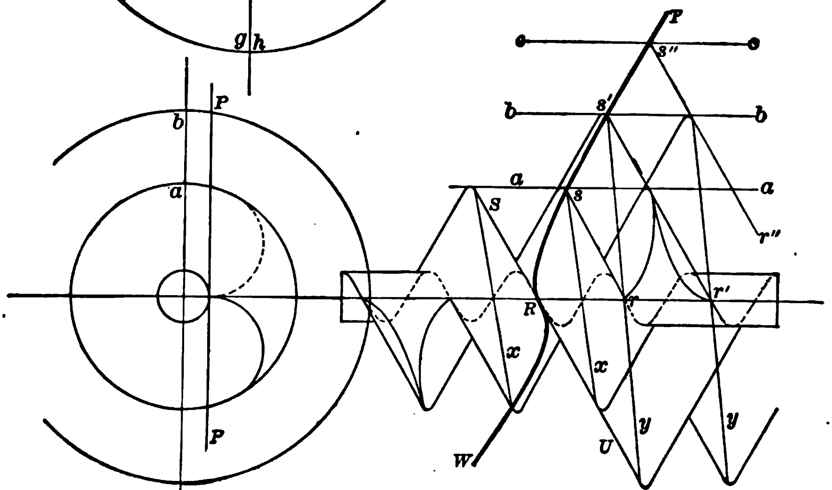


Fig. 224*a*

the gorge helix KCL, but also a serpentine void whose transverse section is of the form of the loop $u_1w_1c_1z_1$.

In these circumstances, a groove of the form $h_1u_1g_1$ can be cut

in the outer cylinder, bounded by helicoidal surfaces of which *DE* is the generatrix: but the actual formation of the surface below the point u_1 is impracticable.

277. In Fig. 223, the angle $d'c'h'$, between *DE* and the vertical element of the inner cylinder through *C*, is nevertheless greater than that between this element and the tangent to the helix *KCL*. Now in Fig. 224, these angles are equal; in other words, the generatrix *DE* is tangent at *C* to the helix: the surface, although an oblique helicoid, is now developable, being in fact the helical convolute previously discussed.

In the meridian section, the loop has now disappeared, u_1 having retreated to c_1, at which point the curve $g_1c_1h_1$ forms a *cusp;* the tangent to this curve at this point, moreover, is horizontal instead of vertical, and the helix *KCL* is no longer a gorge line but an **edge of regression**; along which the helicoid intersects the cylinder, to which it is *not* tangent, as has been erroneously stated.

Finally, in Fig. 225 the angle $d'c'h'$ is less than the angle between the tangent to *KCL* and the vertical element through *C*. We now have again as the meridian section a curve whose tangent at its vertex c_1 is an element of the inner cylinder; and as before the helix *KCL* is a gorge line, along which the helicoid is tangent to that cylinder.

278. Peculiarities of Transverse Sections. The transverse section of any helicoid *may* be determined in the ordinary manner, by finding the points in which any number of elements pierce the cutting plane. But though perfectly correct in theory, this method is in this case, as in many others, practically more laborious and far less reliable than that of constructing the curve by the aid of other known properties.

Thus, let it be required to draw the transverse section by a plane through *C*, in Fig. 224. The axis being vertical, this plane is horizontal, and the required section TcT' is no other than the horizontal trace of the surface; and this (**128**) is the involute of the circular base of the cylinder upon which the helix *KCL* is traced. Drawing tangents at b, l, a, etc., set off upon them $bp =$ arc cb, $lm =$ arc cl, and so on; the curve passing through the points thus located is the required section: the distance rs, set off upon the

most remote tangent, is in this case equal, of course, to the semicircumference of the base.

N. B. This curve forms a cusp at C, as did the meridian section; and so will the section of this particular surface by any other plane

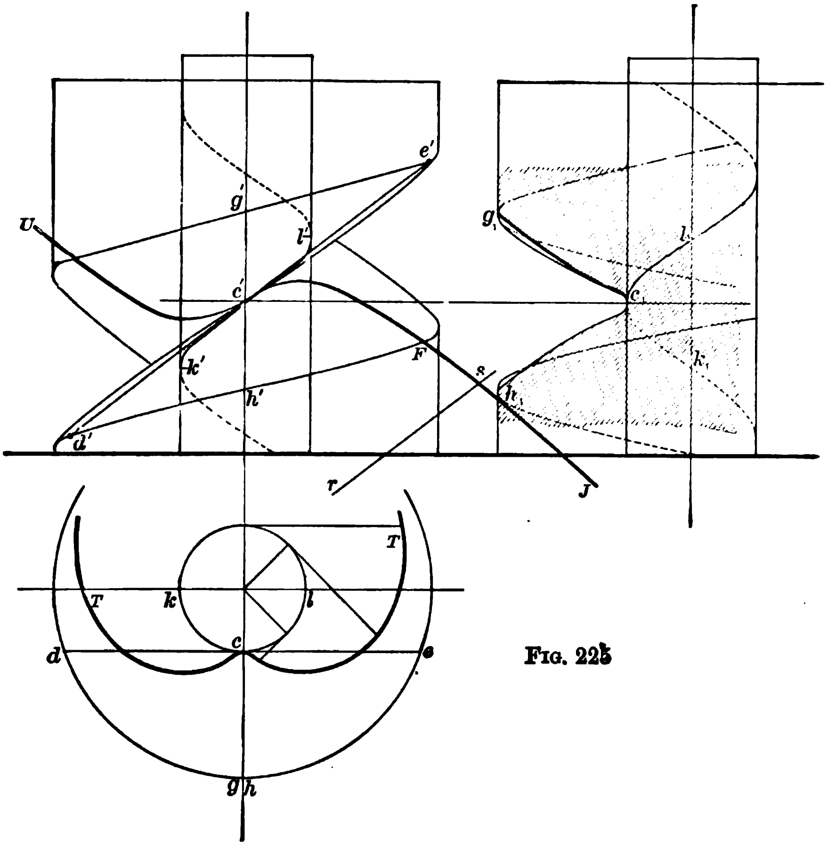


FIG. 225

through the same point, with the exception of those planes which pass through the generatrix DE.

279. The transverse section of the surface shown in Fig. 223 will not be a true involute, because the generatrix is not tangent to the helix KCL; but it may be constructed in a manner analogous to that above described. In Fig. 223a, let $c'l'r'$ be a half turn of the helix; draw through c' a horizontal line $c's'$, and through r' the vertical projection of an element of the helicoid; then $c's'$ is evidently the distance to be set off as rs on the most remote tan-

gent. The motion of the generatrix consists of a uniform rotation, combined with a uniform axial advance; consequently, dividing $c's'$ in Fig. 223a into equal parts by the points 1, 2, 3, and the semi-circumference clr into the same number of equal parts, we draw tangents at the points of subdivision b, l, a, and set off upon them the distances $b1 = c'1$, $l2 = c'2$, $a3 = c'3$: the curve passing through the points 1, 2, 3, s, is the section required. As many intermediate points as may be deemed necessary can be located in like manner; and the resulting curve in this case will be looped, crossing itself at u, the distance uo being equal to u_1o_1 in the meridian section.

280. The formation of this loop in Fig. 223 is obviously due to the fact that the distances $b1$, $l2$, etc., set off on the tangents, are greater than the arcs cb, cl, etc. In Fig. 225, on the other hand, these distances, determined in the same manner, are less than the corresponding arcs. The result is that the curve TcT has a wave instead of a loop: and it is observed that the meridian section also possesses this distinguishing feature.

In thus passing from the loop of Fig. 223 to the cusp of Fig. 224 and the wave of Fig. 225, the angle $d'c'h'$ has been progressively diminished. If on the other hand that angle be increased, the loop will become larger, the curvature at c growing less and less, until when the angle reaches the limit of 90°, as in Fig. 222, there will be no curvature at all, and the transverse section will be simply the generatrix DE itself. If this limit be passed and the angle $d'c'h'$ made obtuse, as in Fig. 221, the section TcT, still tangent at C to that of the inner cylinder, will curve in the opposite direction.

281. Intersections of the Oblique Helicoid with Itself. The plane PP, in Fig. 224a, is tangent to the inner cylinder, upon which lies the helical directrix of the developable helicoid represented by this figure. This plane cuts the surface in the generatrix URS, and also in the curve WRT, as shown in the side view. After one revolution, RS will be found at rs, after another at $r's'$, and so on; and in these new positions will cut the curve RT in the points s, s', s'', etc. The point s describes a helix xx, lying on a cylinder of which aa is an element; the point s' traces the helix

yy, on a cylinder of which *bb* is an element; the point s'' in like manner traces a helix, not shown, on a cylinder of which *cc* is an element—and so on indefinitely; and these helices, all having the same pitch as the original directrix on the inner cylinder, are evidently the lines of intersection between the successive convolutions of the surface.

And it is obvious that as shown in the figure a screw may be cut, the crest of the thread being the helix *xx*, and its root being the directrix: or a larger one of the same pitch, the thread having *yy* for its crest and *xx* for its root; these screws will be *single-threaded*, and their surfaces are portions of the same helicoid.

Fig. 224, on the other hand, represents a *double-threaded* screw; as seen in the side view, two helices, $k'c'l'$, $f'q't'$, are traced on the inner cylinder, to which the two generatrices, $d'c'e'$, $f'u'$, are respectively tangent: these lines thus generate two distinct helicoids, which intersect each other in the helices $d'h'u'$, $v'g'e'$, forming the crests of the threads.

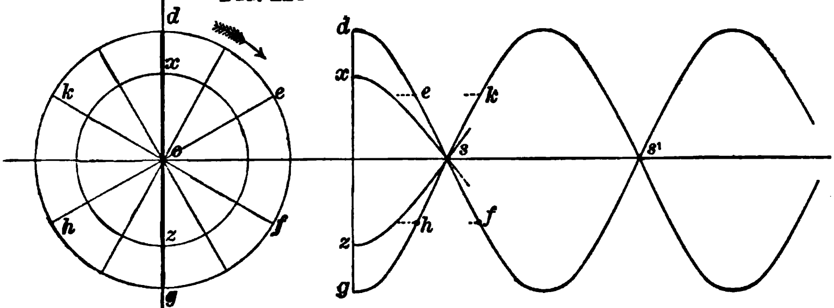
282. That the surface of a warped helicoid will also intersect itself in a series of helices is shown in Fig. 225; a plane through *DE*, parallel to the axis, cuts the surface in the curve $Uc'FJ$, which, however, is not tangent to the generatrix, but crosses it at the point *C*. After one revolution, this generatrix will have the position *rs*, cutting the curve at *s*, which point will describe a helix corresponding to *xx* in the preceding figure; and successive ones may be determined in the manner above explained.

In the case of the right helicoid, however, the plane tangent to the inner cylinder cuts the surface in the generatrix only; and since this is parallel to its position after any given number of complete revolutions, this surface does not intersect itself like the others, no matter how far extended.

283. Special Case of the Right Helicoid. In Fig. 226, the generatrix intersects the axis at right angles, and the rotation being in the direction indicated by the arrow in the end view, the point *d* describes the helix *def*, while *g* describes the helix *ghk*; any other points of the generatrix, as *x*, *z*, will describe helices of different obliquities, since they lie upon cylinders of different diameters, but because they all have the same pitch, these helices will in the side

view intersect in the points s, s', etc., lying on the projection of the axis. This figure represents an ideal single-threaded screw,

FIG. 226



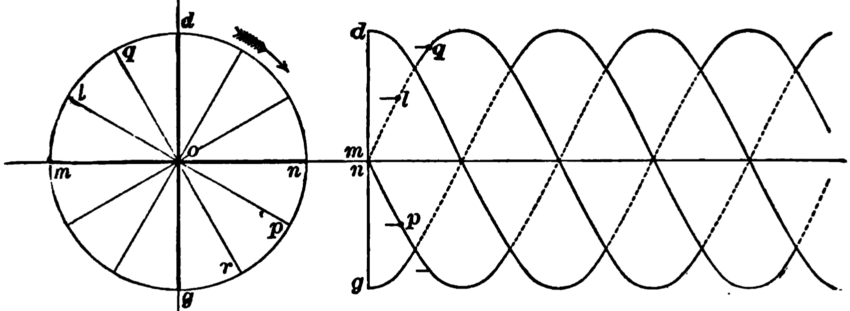


FIG. 227

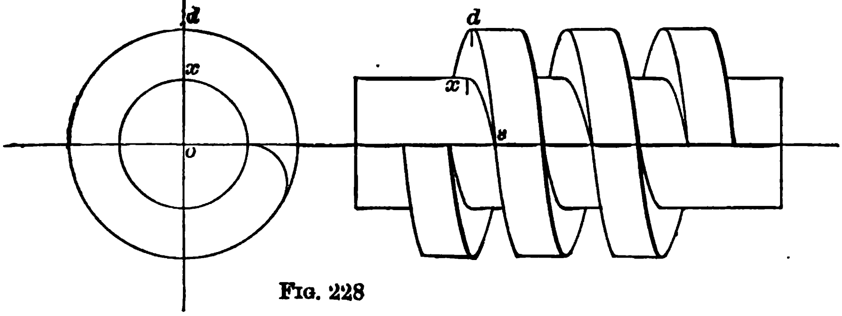


FIG. 228

which may be conceived as being formed by uniformly twisting a thin rectangular strip of flexible metal about its longitudinal centre line. Suppose two such strips to be fastened together at right

angles, presenting when viewed endwise the form of the cross, *dg*, *mn*, Fig. 227; let each be uniformly twisted in the same direction and at the same rate as in the preceding figure: then *m* will describe the helix *mlq*, while *n* describes the helix *npr*, and the result will be the formation of the surface of an ideal double-threaded screw. The practical application is shown in Fig. 228, in the construction of a single-threaded screw, cut upon a cylinder whose radius is *od*, the depth of the groove being limited by the central "core," whose radius is *ox*; upon the surface of this inner cylinder lie the helices corresponding to *xs*, *zs*, in Fig. 226.

284. Special Case of the Oblique Helicoid. In Fig. 229, the generatrix *DCE* cuts the axis acutely at *C*; the helix described by the point *C* therefore coincides with the axis itself. Let $c'm'$ be one half the pitch, then at the end of a half revolution the generatrix will have the position $e'm'f'$, which intersects the first position in *E*; the path of this point, then, is the first of the helices in which the generated surface intersects itself. Set up $m'r'$ equal to the pitch, then at the end of a revolution and a half the generatrix will have the position $r's'$, which cuts $d'e'$ produced in s', whose path $s'x'm'$ is the second of the series of helices, as explained in (**281**).

The visible contour, as in all other cases, is tangent to the projections of all lines of the surface which it intersects (except those whose rectilinear tangents at these points of intersection are perpendicular to the plane of projection); thus, it is tangent to the helix $d'c'e'$, and at l' and n', to the path of the point *K*: again, *uu*, *ww*, are the horizontal projections of two elements, and the contour is tangent at u' and w' to their vertical projections. It is also obviously tangent to the axis at a', the middle point of $c'm'$; this therefore is the vertex of the curve, which is symmetrical with respect to the horizontal line through a', e'. The contour line through o', when produced, as $o'q'$, will also be tangent to the helix $m'x's'$; it is evidently asymptotic to $d'c's'$, and closely resembles the hyperbola having the same vertex and asymptotes; but the latter, as shown by the broken line $o'p'$, lies within the curve under consideration.

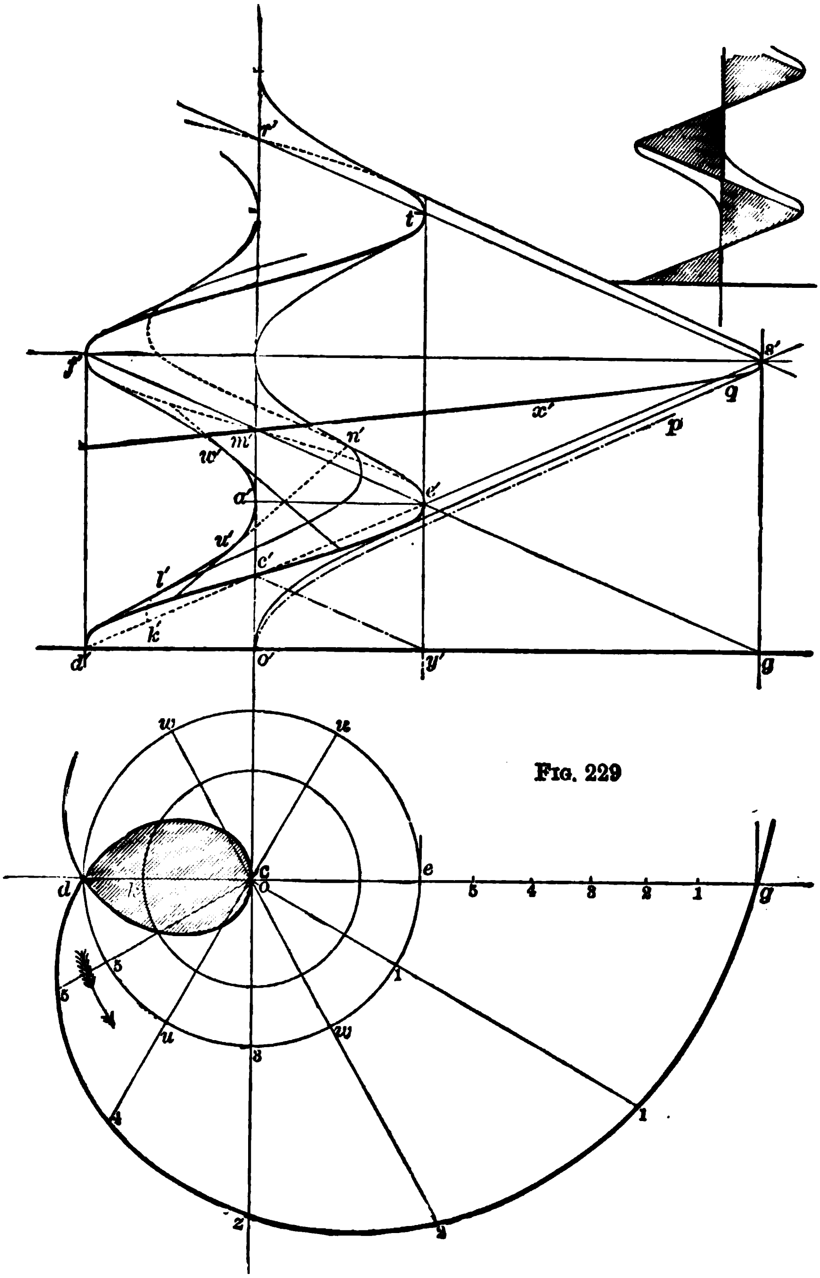


FIG. 229

285. The horizontal trace of this helicoid is an Archimedean spiral. If $c'd'$ be supposed to rotate without advancing, it will after half a revolution occupy the position $c'y'$; but it does meantime actually advance to the position $m'e'$ parallel to $c'y'$, and when produced pierces **H** at G: and since $c'm' = 2o'c'$, we will have $y'g' = 2o'y', = d'y', = de$. Since the revolution and the axial advance are both uniform, the trace is, as stated, an equable spiral, of which $y'g'$, or eg, is the radial expansion in a half revolution. This should be carefully constructed, by dividing eg and the semicircumference dwe into proportional parts, and setting off the distances $c1$, $c2$, etc., on the corresponding radiants: by this means, the horizontal traces of any given elements can be more accurately located than in any other way.

If the surface be limited by the second helix of intersection $m'x's'$, a section by a transverse plane through $f's'$ will be bounded by two symmetrical curves, of which one is the portion dzg of this trace. If the helicoid be limited by the first helix $d'c'e'f'$, the section by the same plane, or by **H** itself, will be the shaded loop of the spiral: the meridian section by a plane parallel to **V** in this case consisting merely of a series of isosceles triangles whose bases coincide with the axis, as shown on a reduced scale in the small detached diagram.

286. Practical Application. This helicoid, as previously stated, forms the surface of the V-threaded screw. It is quite obvious that such a screw may be formed by taking, in Fig. 229, the helix $m'x's'$ for the crest, and the helix $d'c'e'$ for the root, of the thread; both sides of which will then be formed of portions of the same helicoid. But it is not essential that this should be so; and in the majority of practical cases the opposite sides of the thread are portions of two different helicoids. Thus in Fig. 230, the screw may be regarded as being formed by winding a bar of flexible metal, having the triangular section emo, around the central core; the pitch mn being equal to eo. Prolonging me, mo, to cut the axis in c and d, it will be seen that if cd be $1\frac{1}{2}$ times the pitch, it will correspond to $c'r'$ of the preceding figure, md being then simply another position of mc. This, however, is not the case; we have here two independent lines each generating the helicoidal surface

of one side of the thread. These lines are usually, as in this illustration, equally inclined to the axis; which, nevertheless, is by no means essential, since the section *emo* of the flexible bar might have been a right-angled, or a scalene, instead of an isosceles

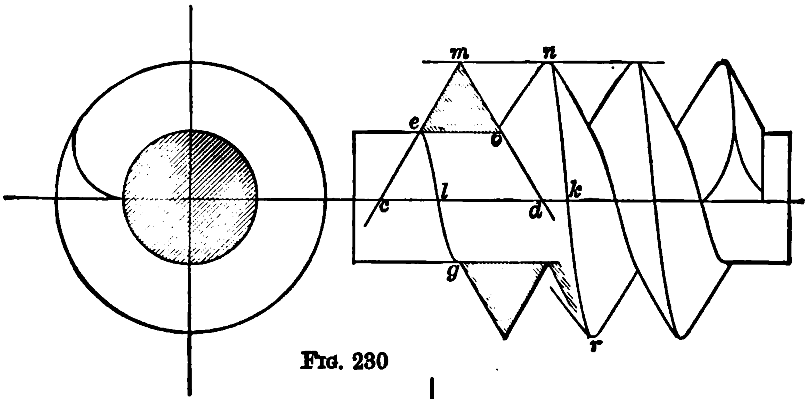


FIG. 230

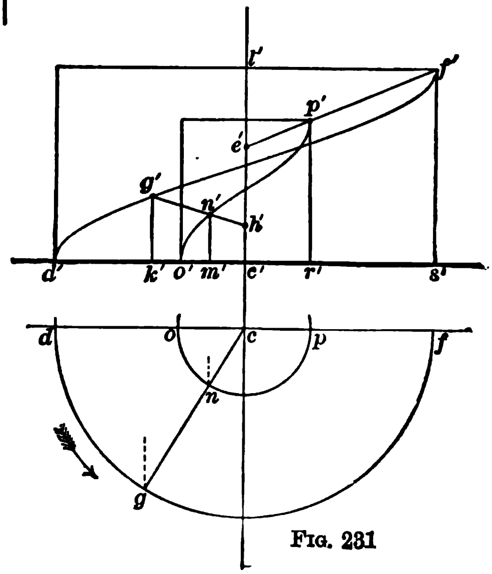


FIG. 231

triangle; and such non-symmetrical screw-threads are sometimes used in practice.

If the pitch be doubled, it is clear that another thread may be formed in the intervening space; if it be trebled, two threads can be added; and so on indefinitely.

For the sake of uniformity the helices have been drawn right-

handed throughout; it need hardly be stated that they might have been made left-handed without in any way affecting the argument: and in practice they often are, in the construction of ordinary screws as well as of screw-propellers.

287. Helicoids of Varying Pitch. If in Fig. 231 the line DC, which is perpendicular to the vertical axis, were to rise in a half revolution to the position $f'l'$, rotating and advancing uniformly, it would generate a right helicoid, the point D tracing the helix DGF. Now let the line rotate uniformly, in contact with this helix and with the axis, the point C at the same time advancing uniformly, but less rapidly than before, so that in a half revolution it reaches the altitude e' instead of l'. By this motion a different surface will be generated, which is a helicoid of varying pitch: since $e'f'$ is longer than $c'd'$, it is evident that the point D does not describe the helix DGF, but since C remains in contact with the axis, there is a sliding of the generatrix upon the guide helix during the motion supposed.

Now draw the inner cylinder, with any radius oc at pleasure; the generatrix in its first position pierces this cylinder at O, and after a half revolution, pierces it at P.

Set off the arc dg, say one-third of the semi-circumference, and draw gc, the horizontal projection of the generatrix in an intermediate position; project g to g' on the outer helix; the altitude $k'g'$ will be one-third of $s'f'$: set up $c'h' = \frac{1}{3}c'e'$, then $g'h'$ is the vertical projection of this line, which pierces the inner cylinder at N.

It is obvious that the altitude $m'n'$ will be one-third of $r'p'$; and since the same argument applies to any other position of the generatrix, it follows that the inner cylinder cuts the helicoid in a true helix ONP, and will do so whatever its radius may be, the pitch, evidently, varying with the radius: and the same applies to exterior cylinders as well.

288. In Fig. 232, the generatrix DC in its first position lies in **H**; let DRF be the guide helix, and FE the position of the generatrix after a half revolution; let mc, nc be the horizontal projections of intermediate positions; if these be revolved about the axis until parallel to **V**, their outer extremities will appear as

m', n', dividing $s'f'$ in the same proportion in which their inner extremities divide $c'e'$. These lines will therefore if prolonged pierce **H** in the same point O; as also will $k'l'$, the position of the generatrix after a revolution and a half—from which it follows that all the elements of this surface pierce **H** at the same distance from the axis. In other words the horizontal trace of the surface consists of the circle whose radius is co, and also of an indefinite right line coinciding with DC, the element which lies in the horizontal plane.

289. The transverse section by any other plane as PP will be a spiral, wux, of which the points u and w are obtained directly, since the elements $g'h'$, $l'k'$, in the vertical projection pierce this plane at u', w'. To find other points, draw intermediate elements, as through m and n in the horizontal projection; these when revolved until parallel to **V** will pass through m'', n'', points dividing $h'k''$ into parts proportional to the divisions of $g'l'$, and will be cut by PP at points whose distances from the axis are to be set off on cm, cn: and in like manner any number of points may be determined, and the curve extended in either direction. This curve is peculiar in possessing a circular as well as a rectilinear asymptote; if continued in the direction uw, it will pass through the pole c, and then again expanding at a decreasing rate, it will after an infinite number of turns be tangent internally to the circle oqt: if continued in the opposite direction, it will be tangent at infinity to a line parallel to cd, and lying at a distance from it equal to the circumference of the circular asymptote.

290. In Fig. 233, the generatrix DC in its first position also lies in **H**, but at a distance cu from the axis; DKF is a helix traced on the cylinder whose radius is ud. Now let DC revolve uniformly about the axis, in contact with this helix, the point C at the same time moving uniformly along the element of contact with the cylinder whose radius is cu, so that after a half revolution the line occupies the position FE. By reasoning similar to that used in connection with Fig. 231, it can be shown that the surface thus generated will intersect any cylinder having the same axis in a true helix; this form of the helicoid is, then, the general one, of which that above discussed is a special case.

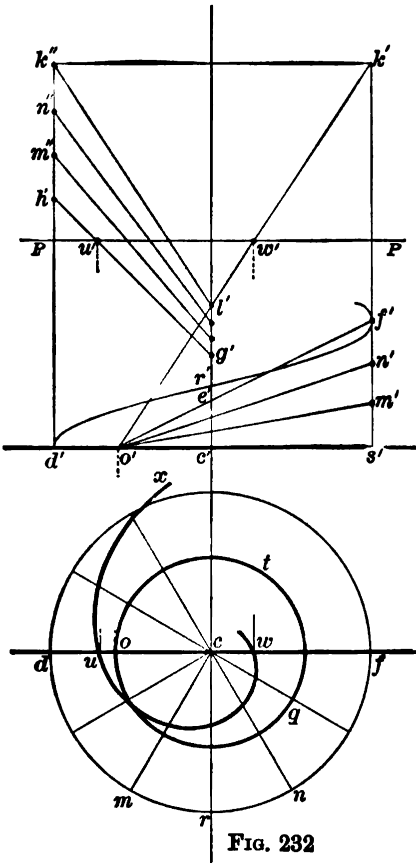


Fig. 232

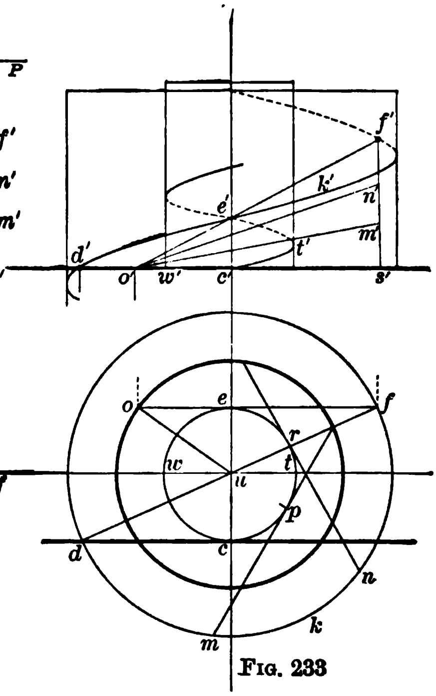


Fig. 233

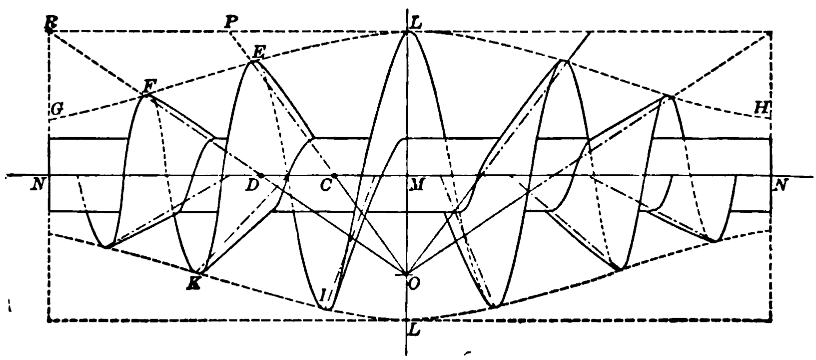


Fig. 234

Drawing *mp*, *nr*, tangent to the circle whose radius is *uc*, and revolving the elements of which these are the horizontal projections until they are parallel to **V**, it may be shown as in Fig. 231 that they will then pierce **H** in the same point *O*: showing that the horizontal trace of this surface consists of the circle whose radius is *uo*, and also of a right line coinciding with *DC*, the original position of the generatrix.

291. Practical Application. When the generatrix intersects the axis, the helicoid of varying pitch is often practically employed, forming the acting surfaces of screw-propellers with what is technically called "radially increasing pitch"; which are swept up, of course, by that portion only of the generatrix which lies on one side of the axis.

To aid in gaining a clear idea of the nature of this surface, there is represented in Fig. 234 a portion of it generated by a line of definite length *ML*; the pitch at that distance from the axis being *LP*, *PR*, while the pitch at the axis itself is *MC*, *CD*. The generatrix *ML* being perpendicular to the axis, will after successive revolutions take the positions *CEP*, *DFR*, etc.; the point *L* thus tracing a *quasi*-helical curve *LIEK*, lying on a surface of revolution whose meridian line is *LEFG*: since, as shown in (**288**), *RD*, *PC*, etc., when produced will meet in the same point *O* on the prolongation of *LM*, the outline of this surface of revolution is the waved branch of a conchoid, of which the pole is *O* and the directrix is *NN* the axis of the helicoid. In order to throw the surface into stronger relief, a concentric cylinder is introduced, which, as before shown, cuts it in a true helix, of which portions are visible.

292. From this it is clearly seen that the surface is divided into two symmetrical parts by the **median plane** *LL*, which contains the element perpendicular to the axis. In the immediate neighborhood of this plane the surface resembles the right helicoid, while at sensible distances from it there is a greater similarity to the oblique helicoid. And it is specially to be noted, that the generatrices incline in opposite directions on the two sides of this plane; so that in the construction of a propeller **it does not suffice to give merely the pitches** at the rim and the hub respectively; since, while the form

of the surface would thus be definitely fixed, the particular part of it to be used would not be located: it is necessary therefore to give in addition the precise inclination to the axis of some specified rectilinear element of the proposed blade.

THE CYLINDROID.

293. The Cylindroid is a warped surface with a plane directer, and is derived from the cylinder in a manner which will be readily understood by the aid of the pictorial representation, Fig. 235. On the left is shown the half of a circular cylinder, with its axis in **H** and its elements parallel to **V**; and this semi-cylinder is cut obliquely by two vertical planes, forming the sections *erd*, *msn*: the elements cut the outlines of these sections in the corresponding points 1 1, 2 2, etc.

Now, the section *erd* remaining fixed, let the other section be moved upward by translation in its own plane through any given distance; the diameter *mon* will then, as shown on the right, be vertically over and parallel to its orginal position, here represented by $m_1o_1n_1$: moreover, the relative positions of the points 1, 2, 3, upon the arc *ns*, will be the same as before.

Next, joining these points with the corresponding points 1, 2, 3, upon the arc *dr*, the new lines 1 1, 2 2, etc., will be the rectilinear elements of the surface under consideration. By construction these elements are parallel to **V**, which therefore is the plane directer in the case here illustrated.

The method of representing this surface in projection is too obvious from the above to require further explanation; nor does the surface itself possess any remarkable features, with the exception that the limiting tangent planes, as *LL* for instance, are tangent all along the elements which they contain.

294. Practical Application. The cylindroid may be used to form the roof of a transverse gallery connecting two parallel arched passages on different levels.

The floor of such a gallery, if constructed on the same principle, will also be a warped surface; it is evident that in the right-hand figure, *mn*, *de*, are the directrices, and *nd*, *me*, are two elements,

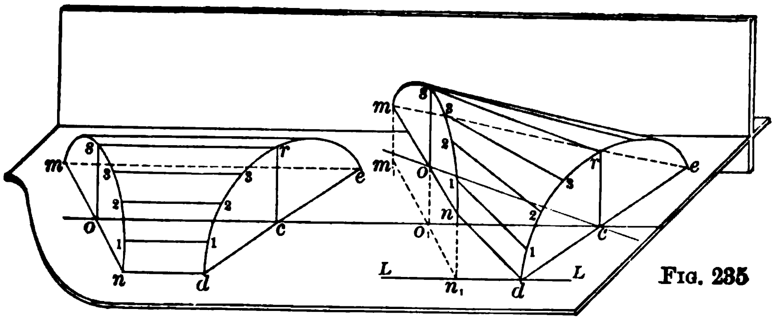


FIG. 235

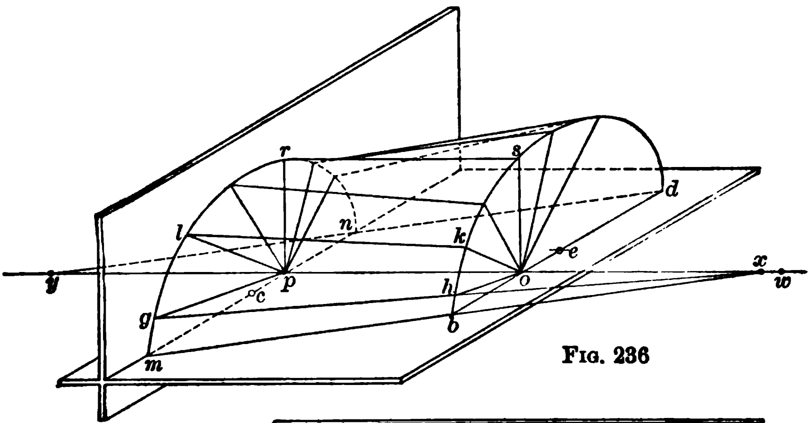


FIG. 236

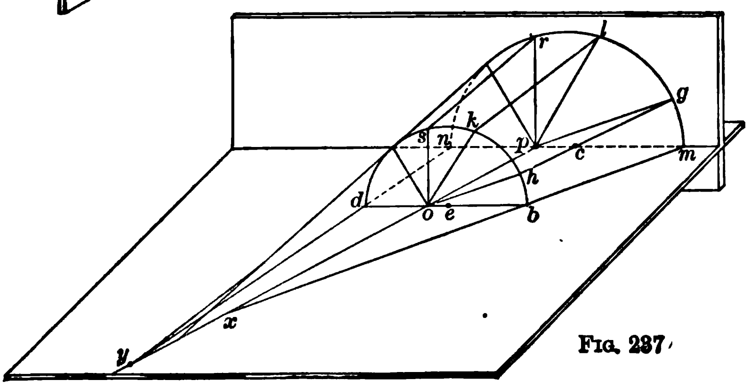


FIG. 237

of a hyperbolic paraboloid, which has **H** for one plane directer and **V** for the other.

In the preceding illustration, the circular cylinder was selected merely for convenience; it is clear that a similar process may be employed, whether the roof of the original arch be circular, elliptical, or of any other section.

THE COW'S HORN.

295. This is a warped surface having three directrices, viz., a right line, and two circles in parallel planes: a plane perpendicular to the latter contains the centres of both circles and also the rectilinear generatrix.

In Fig. 236, the circles *mrn*, *bsd*, are of equal diameters and lie in vertical planes, to which the rectilinear directrix *po* is perpendicular; also, the centres, *c* and *e*, are on opposite sides of *po* and equidistant from it. Under these conditions the surface is symmetrically divided by the vertical plane through *po*; and has a practical application in the construction of the **warped arch.**

Any plane through *po*, evidently, will cut the planes of the circles in parallel lines, as *ok*, *pl*, thus determining an element *lk* of the surface. The element *mb* cuts *po* in *x*; *gh* cuts it in *w*, farther from *o*; *lk* would cut it in a point still more remote, and so on, until, when the cutting plane becomes vertical, *pr* and *os* being equal under the assumed conditions, *rs* is parallel to *po*. The elements beyond *rs* will then intersect *op* on the opposite side of the vertical plane; *dn*, evidently, cutting it at a point *y*, making *py* equal to *ox*.

In Fig. 237, none of the above special conditions are imposed; the two circles are of different diameters, the directrix *po* is not perpendicular to their planes, and the centres *c* and *e* lie on the same side of *po*, but at unequal distances from it; moreover these distances are so chosen that the points *p* and *o* do not divide the radii *cn*, *ed*, in the same proportion. This figure, then, represents a general case of the surface, of which the warped arch is only a special form. All the elements now cut the rectilinear directrix in front of the vertical plane, and at finite distances, since no one of them is parallel to it.

296. The obvious use of such surfaces is in forming the roofs of arched passages; which naturally leads to the selection of circles, ellipses, or other symmetrical curves, lying in parallel planes, for the curved directrices. The construction of the roof requires the use of only one-half of each of these curves, which accordingly is all that is shown in the figures: if we suppose the other half to be added, it is evident that the plane containing the centres of these curves and the rectilinear directrix, will divide the complete surface symmetrically, and in general no other plane will do so. The above pictorial representations show not only the nature of the surface, but the method of determining its elements, more clearly than would its projections, which can readily be drawn without further explanation. Since no plane can be tangent to such a surface along an element, the visible contour will in all cases be a curve, though sometimes a very flat one.

Substantially the same method would be employed were the curved directrices in planes not parallel to each other, not similar to each other, or even were they of double curvature; and indeed it may be said that the Cow's Horn is only a special variety of a general class of warped surfaces, having one rectilinear directrix and two curved ones of any kind whatever.

WARPED SURFACES—GENERAL FORMS.

297. In addition to the preceding, warped surfaces having no specific names are sometimes met with in practical operations. These must necessarily be determined either by two curved directrices and a plane directer, or by three curvilinear directrices; and in representing them, three problems may arise. If there be a plane directer, it may be required to draw an element either parallel to a given line therein, or through a given point on one of the directrices; if there be none, it may be required to draw an element through a given point upon either directrix. We will consider these problems in the order given.

298. I. In Fig. 238, let CD, EF, be the directrices of a warped surface; it is required to draw an element parallel to the line MN lying in the plane directer tTt'.

Analysis. Pass through either directrix a cylinder whose ele-

ments are parallel to the given line. The other directrix will pierce this cylinder in one or more points, through either of which an element of the surface may be drawn parallel to the given line.

Construction. Through any points G, H, K, etc., on CD, draw parallels to MN; these lines pierce the vertical projecting cylinder of EF in the points R, O, L, etc., thus determining a curve vertically projected in $e'f'$, and horizontally projected in e_1f_1. This curve is the intersection of the two cylinders, and cuts EF in the point U, through which is drawn the required element UX, parallel to MN.

299. II. In Fig. 239, let CD, EF, be the directrices, tTt' the

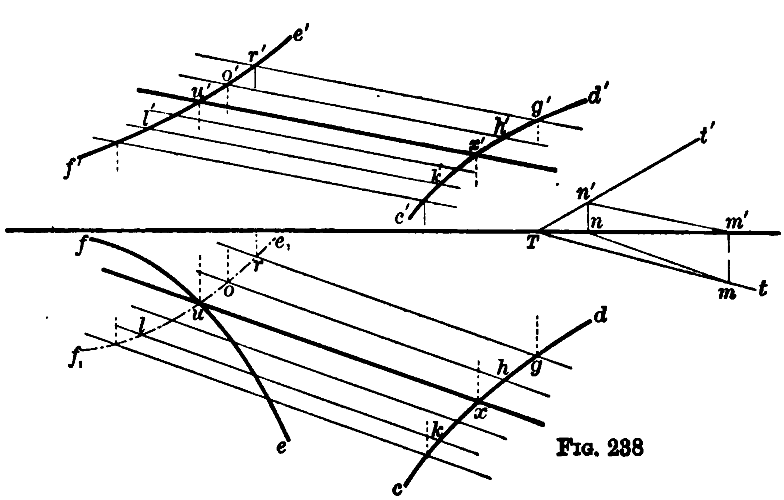


FIG. 238

plane directer; it is required to draw an element of the warped surface, through the point O on CD.

Analysis. Pass through the given point a plane parallel to the plane directer; it will cut the other directrix in a point of the required element.

Construction. Assume in tTt' any point P and also any line MN, and join P by right lines to any points R, S, etc., upon MN. Through O draw parallels to PR, PS, etc.; this series of lines determines a plane parallel to tTt'. This parallel plane cuts the horizontal projecting cylinder of EF in a curve whose projections are ef, e_1f_1; and this curve cuts EF in X, thus determining OX, the element required.

300. III. In Fig. 240, MN, CD, EF, are the directrices of a warped surface; it is required to draw a rectilinear element through the point O on MN.

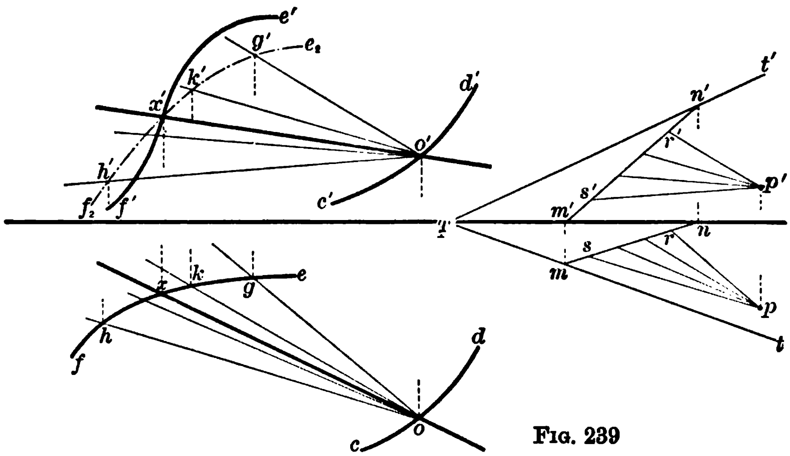


FIG. 239

Analysis. Pass through either of the other directrices a cone of which the given point is the vertex. The third directrix will pierce this cone in one or more points, through either of which and the given point an element of the surface may be drawn.

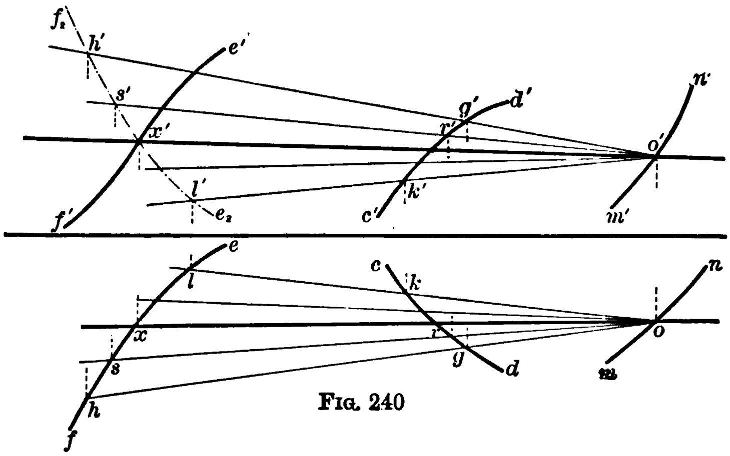


FIG. 240

Construction. Through any points G, R, K, etc., on CD, draw lines from O; these are elements of the cone, and pierce the horizontal projecting cylinder of EF in the points H, S, L, etc.: thus

determining a curve, of which the horizontal projection is ef and the vertical is e_1f_1. This curve is the intersection of the cone with the projecting cylinder, and cuts EF in X, a point of the required element OX.

PLANES TANGENT TO WARPED SURFACES.

301. A plane tangent to any warped surface at a given point may be constructed by the general rule of (**140**), viz.: Draw through the point any two intersecting lines of the surface, and at the point a tangent to each line; these tangents determine the required plane.

The rectilinear element through the given point is one line of the tangent plane, in all cases; and if the surface be doubly ruled, the plane is at once determined by drawing through the point an element of each generation (**142**); as has already been illustrated in the cases of the hyperbolic paraboloid (**236**), the hyperboloid of revolution (**250**), and the elliptical hyperboloid (**260**).

If there be only one set of rectilinear elements, that curve of the surface should be selected to which the tangent can most readily be drawn: the helicoid affords a good illustration, since the tangent to the helix is easily determined.

302. PROBLEM 1. *To draw a plane tangent to an oblique helicoid at a given point.*

Construction. In Fig. 241, let DC, parallel to **V**, be the generatrix, and DFE the helix traced by D: to draw a plane tangent to the surface at the point P upon this helix.

Since the generatrix cuts the axis, set up $c'g' = p'y'$, then $g'p'$ is the vertical projection of the element through P, whose traces are O and S. Since the given helix pierces **H** at D, make pr, perpendicular to cp, equal to the arc pd; then rpm is the horizontal, and $r'p'm'$ is the vertical, projection of the tangent at P to the helix. R is the horizontal trace of this tangent; therefore $torT$ is the horizontal, and $Ts't'$ is the vertical, trace of the required tangent plane.

Note. The horizontal trace of this surface is the Archimedean spiral $kodl$, constructed as in Fig. 229; and it is to be particularly observed that tT is **not** tangent to this trace, because the plane is

tangent to the surface only at the point P, and not along an element. The horizontal trace of a plane tangent to the helicoid at

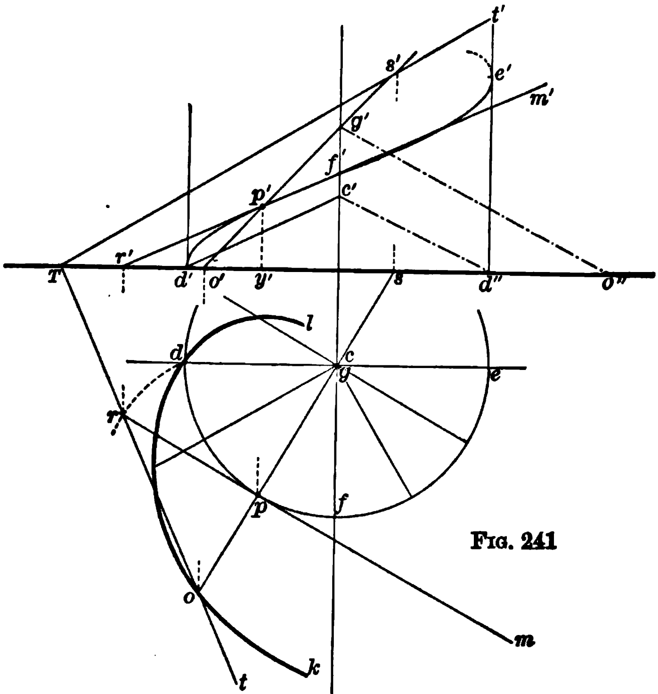


FIG. 241

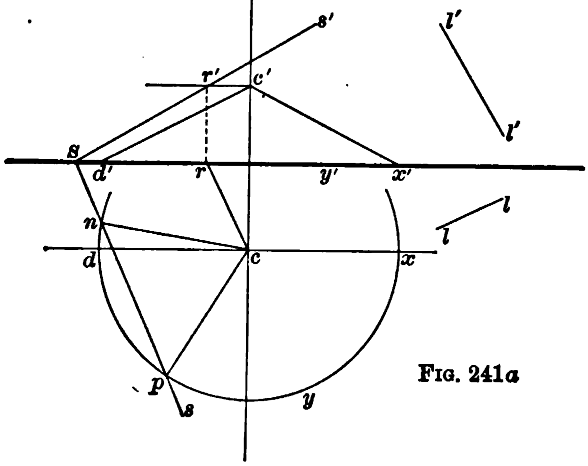


FIG. 241a

D, or any other point in **H**, of course *would* be tangent to this spiral.

303. Problem 2. *To find the point of tangency between a given oblique helicoid and a plane containing a given element thereof.*

Analysis. Since the plane is not parallel to the other elements, it will cut each of them in a point; the curve passing through the points thus found will cut the given element in the required point.

Construction. In Fig. 242, let DC parallel to **V**, be the generatrix of the helicoid, whose pitch is also given. Let GO be the given element of the surface thus determined, and tTt' the given plane.

Having drawn the horizontal trace of the surface, $kodl$, as before, draw cm, cn, etc., the horizontal projections of elements on each side of GO. The points in which these elements pierce tTt can be most readily found by means of a supplementary projection on a plane $zu'z'$, perpendicular to tT. In that projection, the intersection of this plane with the given plane appears as the line z_1z_1', the axis as c_1c_1 perpendicular to HH, and the points m, n, r, are projected at m_1, n_1, r_1, respectively. The rate of axial advance being known, the supplementary projections of the elements through m_1, n_1, etc., are readily determined; they cut z_1z_1' at w_1, e_1, f_1, whence they are projected back to w on cm, e on cn, etc., thus determining the curve wf, which cuts co at p, the horizontal projection of the point sought.

304. Problem 3. *To draw a plane tangent to an oblique helicoid, and perpendicular to a given right line.*

Analysis. Construct the cone directer of the surface, and draw through its vertex an auxiliary plane perpendicular to the given line. If the problem be possible, this plane will in general cut from the cone two elements; these are respectively parallel to those elements of the helicoid, through either of which a plane may be drawn parallel to the auxiliary plane and tangent at some point to the surface. The point of contact is found as in the preceding problem.

Construction. In Fig. 241a, C is the vertex and DYX is the base of the cone directer, LL is the given line, sSs' is the auxiliary plane perpendicular to it, and cp, cn, are the horizontal projections of the two elements: had the plane been tangent to the cone, it is

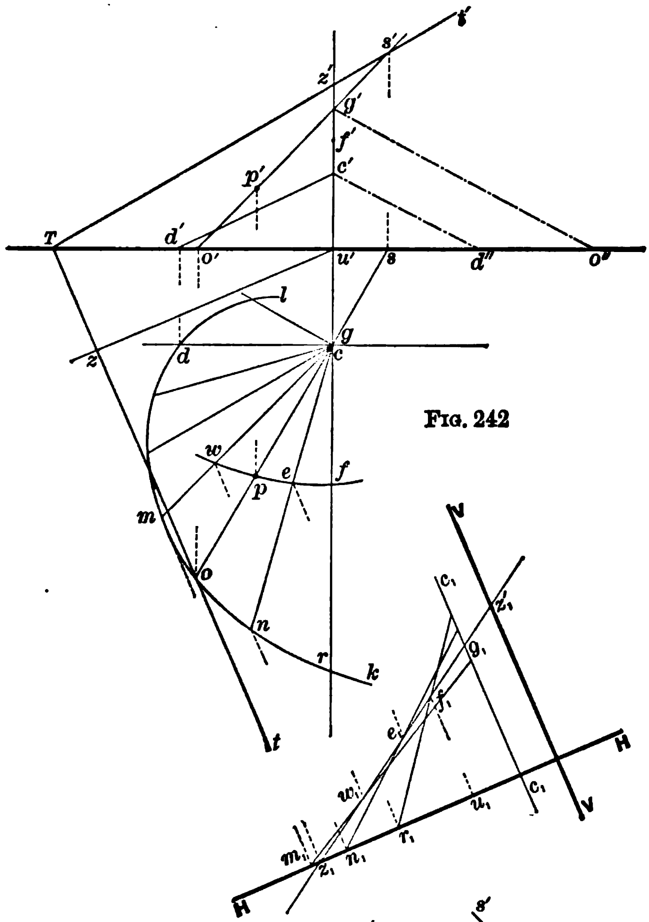


Fig. 242

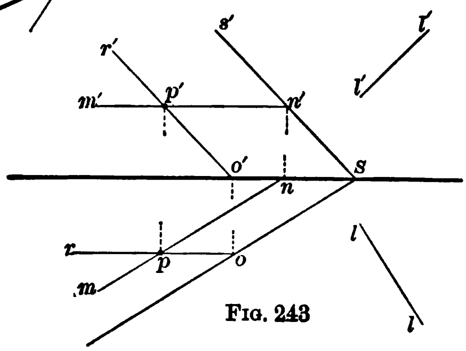


Fig. 243

evident that only the elements of the helicoid parallel to the line of contact, could have planes drawn through them which would satisfy the assigned condition.

Note. The cone directer here drawn, is that of the helicoid represented in Figs. 241 and 242; in each of which also the tangent plane is parallel to sSs' of this diagram, and contains the same element parallel to CP; which facilitates a comparison of the constructions.

305. The same general method, evidently, may be employed in dealing with any warped surface having a cone directer, in regard to which the preceding problem may be proposed. In relation to the hyperboloid of revolution and the elliptical hyperboloid, it is to be noted that since they are doubly ruled, the two elements parallel to those cut from the cone directer at once determine the tangent plane, and their intersection determines the point of contact. If the auxiliary plane be tangent to the cone directer, the two elements of the surface parallel to the line of contact, will in the transverse section be tangent to the gorge curve on opposite sides, and the point of contact will be infinitely remote.

306. PROBLEM 4. *To draw a plane tangent to a hyperbolic paraboloid, and perpendicular to a given right line.*

Argument. This may be accomplished by the following series of operations, viz.:

1. Draw the plane directers and find their line of intersection.
2. Draw a plane perpendicular to the given line.
3. Find the intersection of this plane with each plane directer.
4. Find an element of the surface parallel to each of these lines of intersection (**226**).

These elements will be of different generations, will determine a plane perpendicular to the given line, and will intersect in the point of contact.

307. PROBLEM 5. *To draw a plane tangent to any warped surface, and perpendicular to a given right line.*

Preliminary. In Fig. 243, let LL be the given line, sSs' the plane perpendicular to it. Through any point P in this plane, draw MN parallel to the horizontal trace, and RO parallel to the

vertical trace. The following is then applicable to any warped surface whatever.

Argument. 1. Draw a series of sections of the given surface by horizontal planes; draw a tangent to each section, parallel to MN, and find the point of contact. The line joining these points is the curve of contact between the warped surface and a cylinder whose elements are parallel to MN.

2. Draw a series of sections of the given surface by planes parallel to **V**; draw a tangent to each section parallel to RO, find the point of contact, and draw a curve through the points thus found; this is the line of contact with a second cylinder whose elements are parallel to RO.

3. The two curves thus determined will intersect in the required point of contact between the given surface and a plane perpendicular to the given line: which plane will of course contain the rectilinear elements of the surface which pass through the point.

Note. The lines MN, RO, have in the above argument been made parallel to **H** and **V** respectively, for convenience only; it is clear that any other lines in sSs' might have been used as well, the elements of the two tangent cylinders being drawn parallel to them: but in general the execution would be more laborious.

308. PROBLEM 6. *To draw a plane tangent to any warped surface, through a given right line.*

Argument. Let the given line be produced until it pierces the surface; then the rectilinear element through the point of penetration, and the given line itself, determine a plane which in general will be tangent to the surface at some point of the element.

If two elements pass through the point, each will determine a tangent plane; so again, if the given line pierce the surface in more than one point, there will be more than one tangent plane. If the given line be parallel to a rectilinear element, the point of penetration will be infinitely remote, and the tangent plane is determined by that element and the given line itself. The point of tangency is, in all cases, the intersection of the rectilinear element with the curve, if there be one, cut from the warped surface by the tangent plane.

There are cases in which there is no such curve—for instance, the given line may lie in a plane tangent to a conoid or cylindroid along an element;—or, the plane determined by the given line and an element of a hyperbolic paraboloid, may be parallel to a plane directer.

309. The problems of passing a plane tangent to a warped surface, and either parallel to a given right line, or through a given point without the surface, are indeterminate.

In the first case, by making a series of sections of the surface by planes parallel to the given line, and drawing a tangent to each, also parallel to it, a cylinder may be constructed, tangent to the surface. Any plane tangent to this cylinder satisfies the conditions.

In the second case, a series of sections of the warped surface being made by planes containing the given point, let a tangent be drawn to each through that point. These tangents are elements of a cone tangent to the given surface, whose vertex is the given point: and any plane tangent to this cone satisfies the conditions.

310. Use of Auxiliary Surfaces. The operation of drawing a plane tangent to a warped surface may sometimes be facilitated by the use of another warped surface. It will presently appear that, as stated in (**144**), one such surface may be tangent to another one all along an element. In that case, any plane tangent to either is tangent to both, if the point of contact lies on the common element; and as has already appeared, it may be much easier to draw a plane tangent to the one than to the other.

TANGENCY OF WARPED SURFACES.

311. Two warped surfaces are tangent to each other, like any others, when they have at any point a common tangent plane. In order that they may be tangent all along a common element, they must have a common tangent plane at every point thereof. And this will be the case, if that condition be satisfied for **any three points** of the given element.

For, in Fig. 244, let LL be an element common to two given warped surfaces, which have a common tangent plane at each of the three points A, B, C. Any intersecting planes passed through these points will cut from one surface the three curves D, E, F,

from the other the curves G, H, I, and from the tangent planes the right lines R, S, T.

The curves D, G, being tangent to each other, have two consecutive points in common; and the same is true of the other pairs, E, H, and F, I. Consequently, if LL be moved either upon D, E, F, or upon G, H, I, as directrices, into its consecutive position, it will lie in both surfaces; which, therefore, have two consecutive rectilinear elements in common. Any plane cutting these, evidently, will cut from the surfaces two lines which have two consecutive points in common, or in other words are tangent to each other: the two surfaces, then, are tangent all along LL.

312. If the two surfaces have a **common plane directer**, and

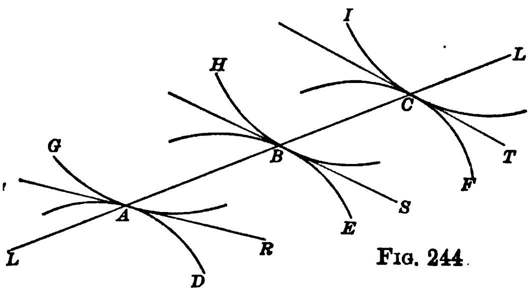


FIG. 244.

common tangent planes at **two** points upon a common element, they will be tangent all along that element. For in this case the motion of LL in Fig. 244 would be completely determined by two of the pairs of curves there shown; and the argument is otherwise the same as above.

313. If in the same figure the generatrix LL be moved upon the tangent lines R, S, T, as directrices, instead of upon either set of curves, it will generate a third warped surface, tangent, all along the element, to both the others: this surface having three rectilinear directrices, must be either a hyperbolic paraboloid, or a warped hyperboloid. If the three intersecting planes passed through A, B, and C are parallel, then R, S, and T are all parallel to one plane, and the surface is a hyperbolic paraboloid; if they are not, the surface will be either a circular or an elliptical hyperboloid. The relations and directions of the intersecting planes

being entirely arbitrary, any number of sets of planes, parallel or not, may be drawn through those points; consequently, any number of hyperbolic paraboloids, and any number of warped hyperboloids, may be constructed, all of which shall be tangent along the element LL to both the given surfaces.

314. Since the directions of the tangents R, S, T, are determined by either of the two given surfaces independently of the other, it might seem that the above would be true of any element of any warped surface. But there are exceptions. It has been seen that a plane may be tangent all along certain elements of a warped surface, as in the cases of the cylindroid and some forms of the conoid. To such a surface, evidently, no other warped surface can be tangent along those elements, except one which possesses the same peculiarity, and this is conspicuously not the case with the hyperboloids: single-curved surfaces, however, may be so, of any kind and of any number.

The normals at all points of such an element, being perpendicular to one plane, are parallel to each other, and thus determine a plane normal to the surface all along the element.

315. The normals to a warped surface at various points of a given element are not in general thus perpendicular to any one plane. But if they are not, then, whatever the nature of the given surface, these normals are elements of **a rectangular hyperbolic paraboloid.**

This may be shown as follows: In Fig. 245, let D, E, F, be the curves cut from a warped surface by planes perpendicular to the element LL at the points A, B, C; let X, Y, Z be the normals at those points, which will be perpendicular to the tangents R, S, T lying in the same intersecting planes. If LL be moved upon the tangents to any new position MN, it will, as has already been seen, generate a hyperbolic paraboloid; of which one plane directer will be any plane V perpendicular to LL, and the other will be any plane H parallel to both MN and LL, and consequently perpendicular to V. This tangent paraboloid, then, is rectangular; and if it be revolved about LL through an angle of 90°, the elements R, S, T will coincide with the normals X, Y, Z, the line MN taking the position UW. The paraboloid now has for one set of

elements the series of normals, and for one plane directer the plane V to which they are all parallel; the other plane directer will be any plane P parallel to both UW and LL, and therefore perpendicular to V: and since the revolution was through an angle of 90°, this plane P will also be perpendicular to H.

Note. In this illustration, the conditions have for the sake of clearness been so chosen that the plane directers V, H, P, are respectively parallel to the vertical, horizontal, and profile planes; and for further elucidation, the positions of the tangents R, S, T,

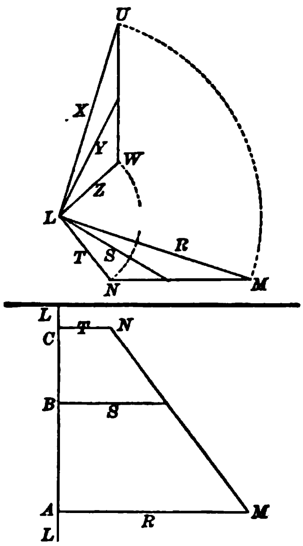


Fig. 246

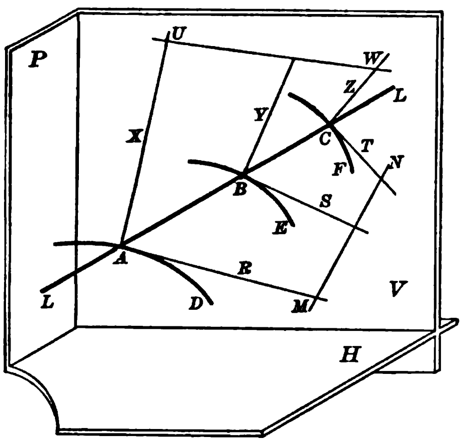


Fig. 245

and the line MN, before and after revolution, are represented by their projections, in Fig. 246, of which no explanation is needed.

316. Applications of the Preceding. The hyperbolic paraboloid is readily drawn, and a plane tangent to it easily determined; it is therefore natural that this surface should be the one usually employed as an auxiliary, in constructing tangent planes to other and more intractable warped surfaces, as suggested in (**310**): the two following examples illustrate its use for this purpose.

317. Problem 1. *To draw a plane tangent to the Cow's Horn at a given point of the surface.*

Construction. In Fig. 247, X_1, Y_1, parallel to **V**, are the circular directrices, whose centres X and Y, as well as the rectilinear directrix EN, lie in the plane JJ parallel to **H**; it is required to

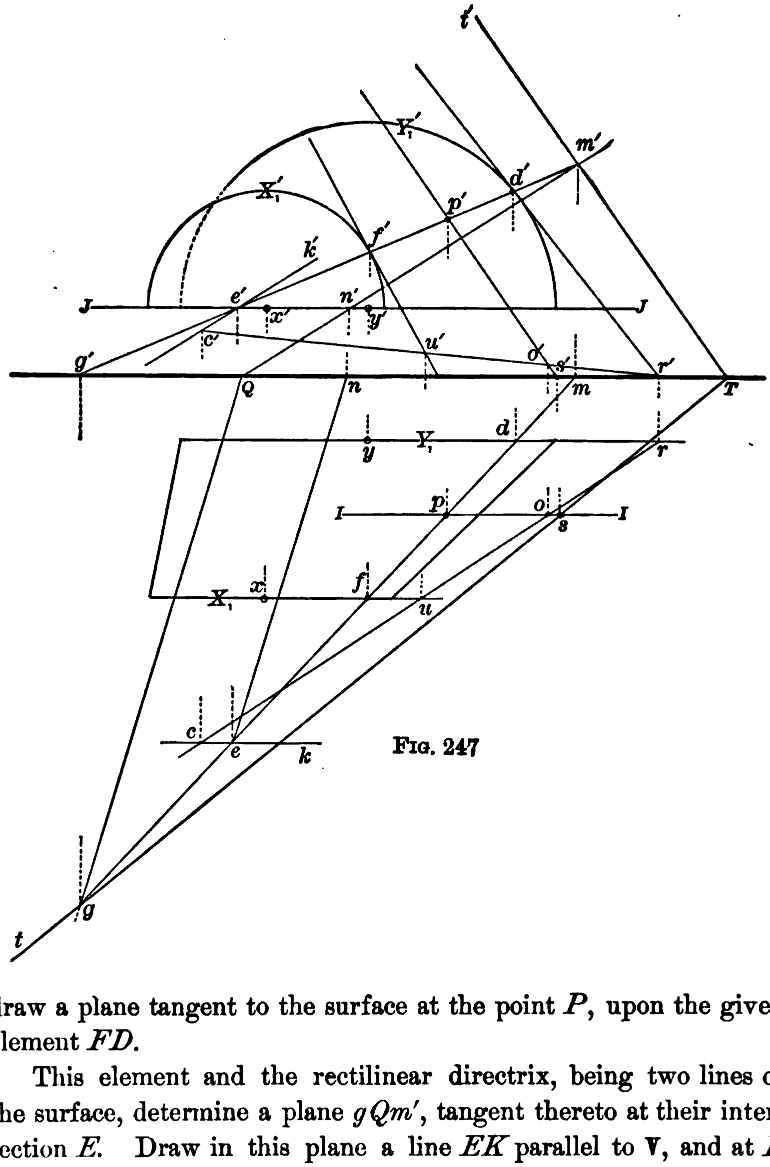


FIG. 247

draw a plane tangent to the surface at the point P, upon the given element FD.

This element and the rectilinear directrix, being two lines of the surface, determine a plane gQm', tangent thereto at their intersection E. Draw in this plane a line EK parallel to **V**, and at D and F draw tangents to the circular directrices: these three lines

are elements of one generation, and ED is an element of the other generation, of a hyperbolic paraboloid having **V** for one plane directer.

Through any point R of the tangent at D, pass a plane containing the tangent at F; this plane cuts EK in C, and CUR is an element of the paraboloid.

Through P pass the plane II parallel to **V**, cutting CR in O, then PO and ED determine a plane tangent at P to the auxiliary paraboloid and therefore to the given surface. ED pierces **H** in G, and PO pierces it in S; also ED pierces **V** in M: consequently tT, the horizontal trace of the required plane, transverses g and s, and Tt', the vertical trace, passes through m'—being, moreover, parallel to $p'o'$ the vertical projection of PO.

318. PROBLEM 2. *To draw a plane tangent to the cylindroid at a given point of the surface.*

Construction. In Fig. 248, let Y, Z, be the circular directrices, **V** the plane directer, and P, on the element CD, the given point. Draw, at C and D, tangents to the circles; these tangents are the directrices, and CD is the generatrix, of the auxiliary hyperbolic paraboloid. The tangents pierce **V** in F and E respectively, and EF is another element of the auxiliary surface. Pass through P a plane parallel to DE and CF; its vertical trace is $r's'$, which cuts $e'f'$ in o', horizontally projected in o on **AB**. Then PO and CD determine the required tangent plane; whose vertical trace is parallel to $c'd'$ the vertical projection of CD.

319. The normals to any warped surface at points of a given element thereof, determine, in general, a hyperbolic paraboloid (**315**), and belong to the same generation. If any element of the other generation of this paraboloid be taken as an axis, the given element by revolving around it will generate an hyperboloid of revolution, which will be tangent to the given surface all along the element: of which the following exhibits a useful application in mechanism.

320. PROBLEM 3. *To construct two hyperboloids of revolution, tangent to each other along an element.*

Construction. In Fig. 249, let the axis of one hyperboloid be vertical, c being its horizontal and $o'c'$ its vertical projection; let

co be the radius of its gorge circle, and OP, parallel to **V**, its generatrix: the projections of this surface are then drawn as in Fig. 209. The gorge radius of which o' is the vertical and *co* is the horizontal projection, is evidently normal to the surface; the normal at P must lie in a plane perpendicular to OP, therefore $c'p'$ perpendicular to $o'p'$ is its vertical and *cp* is its horizontal projec-

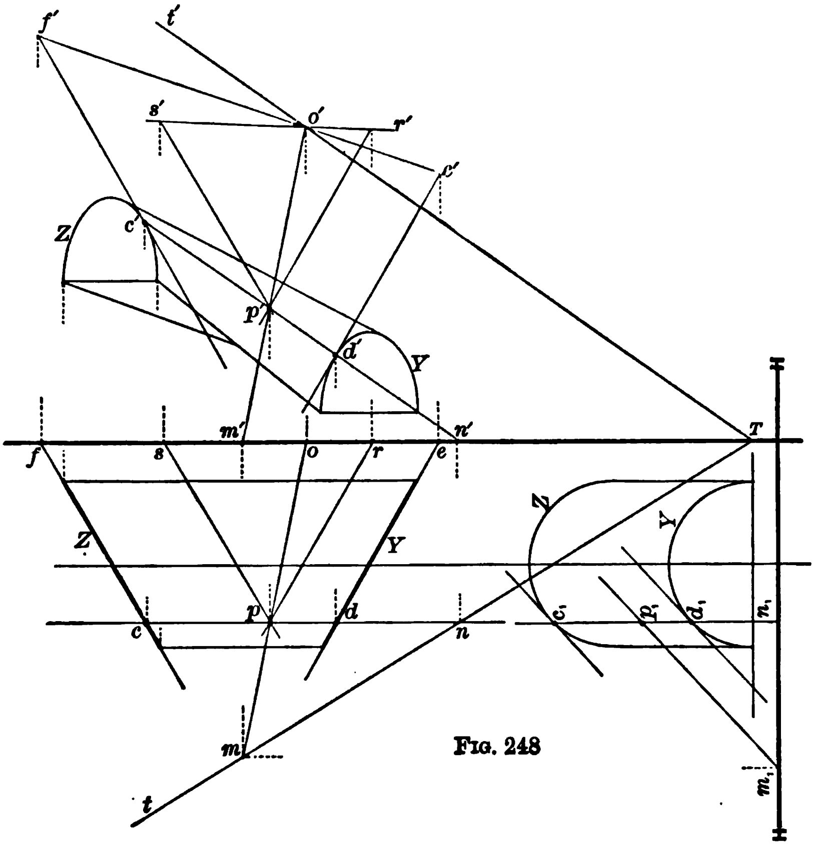


FIG. 248

tion: these normals are elements of one generation, and the vertical axis and the generatrix OP are elements of the other generation of the normal hyperbolic paraboloid,—of which one plane directer is **V**, and the other is perpendicular to OP. Any plane as *JJ* parallel to **V** is seen in the horizontal projection to cut *co* produced, in *e*, and *cp* produced, in *d*; *e* is vertically projected in o', and *d*

in d', on the prolongation of $c'p'$, therefore ed is the horizontal, and $o'd'$ the vertical, projection of another element of the normal paraboloid: which may be taken as the axis of the second hyperboloid.

321. Make a supplementary projection on a plane perpendicular to this axis, looking in the direction of the arrow w. In this view,

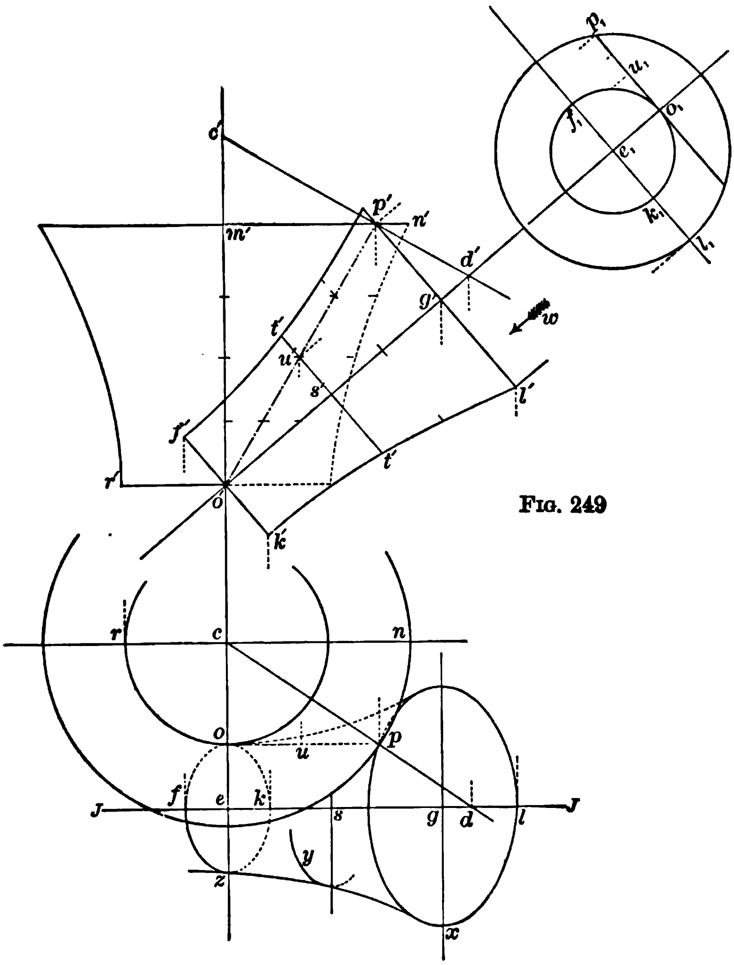


FIG. 249

eo is seen in its true length as e_1o_1, and the common element OP as o_1p_1 tangent at o_1 to the circle of the gorge, which in the vertical elevation appears as $f'k'$ perpendicular to $o'd'$. The radius of the upper base, passing through P and also perpendicular to the

inclined axis, is e_1p_1', to which accordingly $g'l'$ in the vertical projection is made equal. Any point U on OP, is projected to u_1 on o_1p_1', and $s'u't'$ perpendicular to $o'd'$ is equal to e_1u_1; and in like manner any desired number of points on the required hyperbola may be found.

In the foreshortened horizontal projection of the inclined hyperboloid, both the gorge circle and the upper base appear as ellipses; as will also any intermediate transverse sections. A portion of one such section through S, is projected at y; the visible contour zx is **the envelope** of all these ellipses, and not, as sometimes supposed, a curve through the extremities of their major axes: in fact it passes through the extremity of only one of them, viz., that of the gorge circle at z, at which point the contour line has a tangent perpendicular to ez the gorge radius.

INTERSECTIONS OF WARPED SURFACES.

322. The intersection of any warped surface with a plane may be determined by finding the points in which its elements pierce the plane. Many such intersections have already been illustrated; and in any case the problem is simple in principle, though the necessary repetition of the same process may render it tedious in execution.

The intersection with any other surface may be determined by the general method, of passing a series of auxiliary planes cutting both surfaces, and joining the points in which the lines cut from each surface intersect each other. Just what system of auxiliary planes will be most convenient, must in the nature of things depend largely upon the peculiarities of any given case, and be decided by the judgment of the operator. Attention to the following points may, however, sometimes be of service;—

1. *If one of the given surfaces be a cylinder;* Planes may be passed through the elements of the warped surface, parallel to those of the cylinder.

2. *If one of the surfaces be a cone;* Planes may be passed through the elements of the warped surface and the vertex of the cone.

3. *If both surfaces are warped, but have a common plane di-*

recter; A system of planes parallel to this plane directer may be used.

In either of these cases, the auxiliary planes will cut right lines from both surfaces; but it does not necessarily follow that these will give the most satisfactory determinations, since they may intersect each other very acutely.

323. The intersection of a helicoid with a surface of revolution having the same axis, is of special interest as being frequently met with in the construction of screw-propellers; a few illustrations of it are therefore appended. That particular form of the helicoid only is here considered in which the generatrix cuts the axis; because in practice it is used, if not exclusively, at least more extensively than any other.

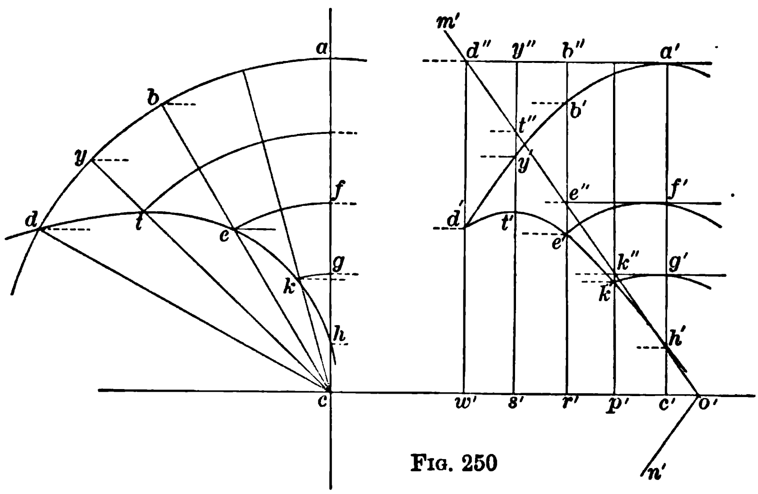


FIG. 250

324. EXAMPLE 1. *Intersection of a helicoid with a right circular cone having the same axis.*

Construction. In Fig. 250 are given, on the left an end view, on the right a side view, of a portion of a right helicoid; *ad*, *a′d′*, being the helical directrix. In the end view, *ca*, *cb*, *cd*, etc., equidistant radii, represent elements; if these be revolved into the vertical plane *ca*, they will in the side view appear as the equidistant lines *c′a′*, *r′b″*, *w′d″*, etc., perpendicular to the axis; these pierce the cone *m′o′n′* at the points *h′*, *e″*, *d″*, etc. Set off on the radii

in the end view **the true distances of these points from the axis, as** $ch = c'h'$, $ce = r'e''$, $ct = s't''$, etc. Project the points thus located, back to the elements in the side view, as d to d', t to t', e to e', etc.; the curves deh, $d'e'h'$, thus determined, are the required projections of the intersection.

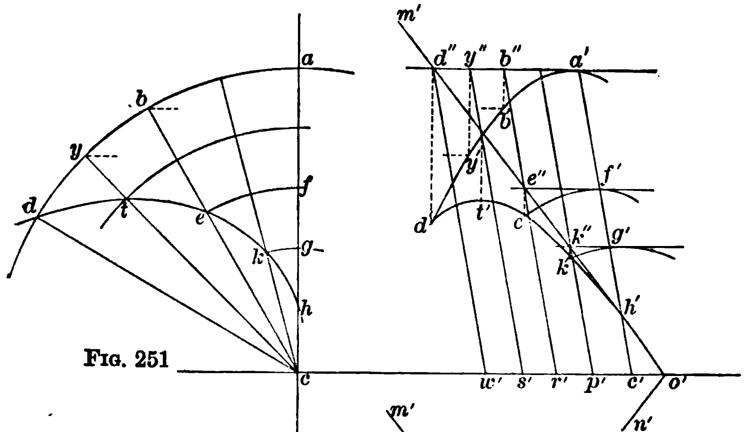


FIG. 251

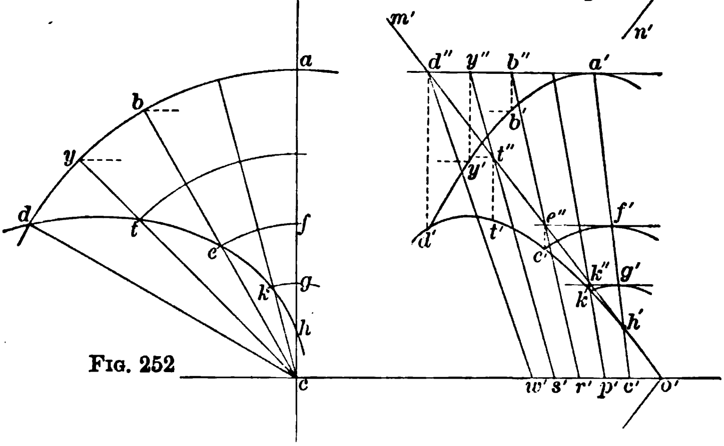


FIG. 252

In the case of the oblique helicoid, Fig. 251, the elements when revolved into the vertical plane ca, appear in the side view as equidistant parallels inclined to the axis.

The construction is the same as before, with the exception that the points d', t', e', etc., are located, not upon the elements, but upon perpendiculars to the axis from the points d'', t'', e'', etc.; because each point must revolve in a plane perpendicular to the axis.

Fig. 252 represents the intersection of the cone with a helicoid of varying pitch. In this case the revolved elements appear in the side view as lines of different inclinations, dividing into the same number of equal parts the distances $a'd''$, $c'w'$; which distances are the same fractions of the pitches at the outer circumference at the axis respectively, that the arc ad is of the whole circumference: otherwise the construction is the same as in Fig. 251.

325. Note. The arcs kg, ef, etc., in the end views, represent concentric cylinders, which cut all these helicoids in true helices—two of them shown in the side views as $k'g'$, $e'f'$. The outlines

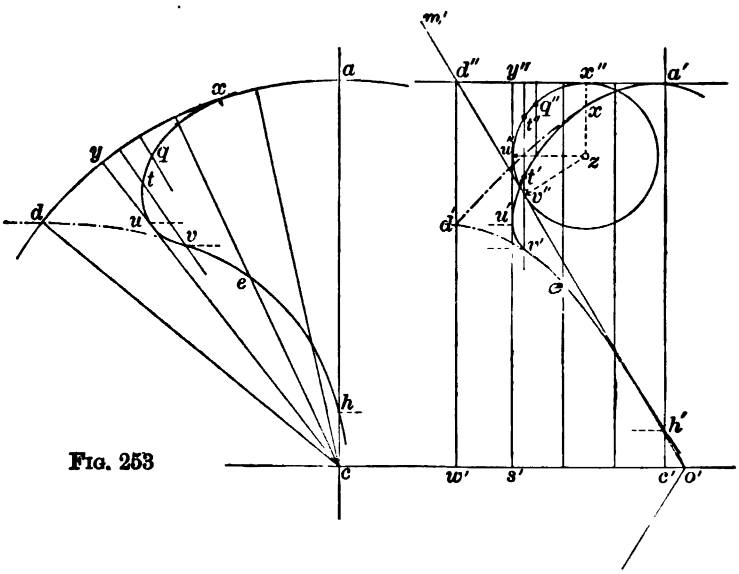


FIG. 253

$k''g'$, $e''f'$, of these cylinders must pass through the points k'', e'', t'', etc., in all three side views; in the first two these points are equidistant, therefore in the end views the points h, g, f, are also equidistant, and the curve deh in Figs. 250 and 251 is an Archimedean spiral: but in Fig. 253 this is not so, for the distances hg, gf, etc., are not equal, but continually increase as the points h, g, f, recede from the centre.

This problem is encountered in determining the form of the trailing edge of a propeller-blade having what is technically called an "overhang"; when it will often be found more satisfactory to

ascertain the distances ck, ce, etc., by calculations based upon the law of the spiral, since accuracy in the end view is most essential.

326. EXAMPLE 2. *Intersection of a helicoid with an annular torus having the same axis.*

Explanatory. A propeller-blade fashioned as above would revolve within a surface of revolution whose meridian section is $o'd''a'$ in either of the three preceding figures: it would also have an objectionable sharp corner at d, d'. The imaginary "box" within which the blade revolves is therefore sometimes 'rounded off at the angle,' as by the arc $x''u''v''$ in Fig. 253. This arc is a part of the circumference of a circle whose centre is z; and the complete circumference is the meridian section of an annular torus having the same axis as the helicoid.

Construction. The warped surface in this figure is a right helicoid, and the mode of operation is substantially the same as in Fig. 250. Thus, dividing the arc ad and the distance $a'd''$ in the same proportion, the elements are represented by radii in the end view and by perpendiculars to the axis in the side view, drawn through the points of division; and the curves deh, $d'e'h'$, are found as in (**324**). The section of the torus is tangent at u'' to the element $s'y''$; set off on cy, the distance $cu = s'u''$, and project u back to $s'y''$ at u'; this determines a limiting point of the curve in each view. The circular section is tangent to $o'm'$ at v'', and an element through this point cuts the circumference also at t'': the distances of these points from the axis being set off on the corresponding radius, the two points t and v are determined—of which the latter is the point of tangency between the spiral deh and the new intersection vtx. By repeating this process as many points as are deemed requisite may be found, and the entire curve of penetration constructed: only that portion is here shown which would form part of the contour of the blade of a propeller.

327. EXAMPLE 3. *Intersection of a helicoid with a cylinder whose elements are parallel to the axis.*

Explanatory. The form of a propeller blade is sometimes fixed by the condition that it shall present a given outline in the end view: the drawing of the other views then involves the problem above mentioned.

Construction. In Fig. 254, let the warped surface again be a right helicoid, and let *mon* be the base of the cylindrical surface.

The operation is simply the converse of the preceding; the points in which the elements of the helicoid pierce the cylinder are seen directly in the end view, and are projected to the corresponding elements as seen in the side view,—as e to e' on $w'd''$, o to o' on $s'y''$, etc. Then in order to find the outline of the surface within

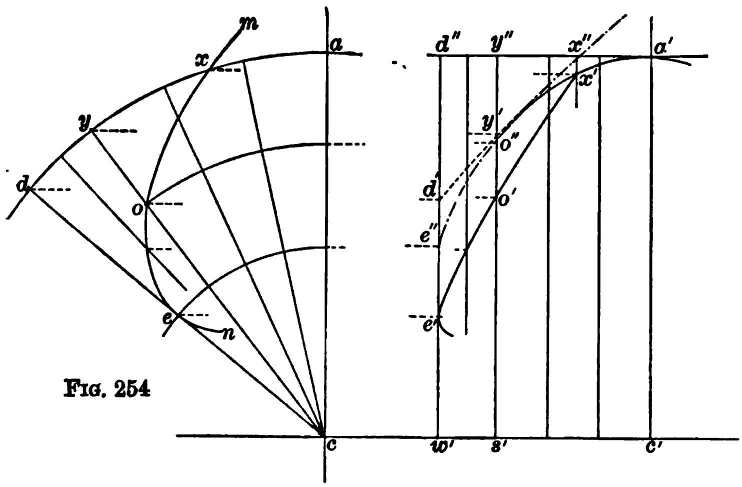


Fig. 254

which the blade revolves, set up $w'e'' = ce$, $s'o'' = co$, and so on, thus determining the required curve $e''o''x''$.

328. These last problems have been illustrated only in connection with the right helicoid, merely for the sake of simplicity. But by attention to the explanations given in (**324**), there will be no difficulty in dealing in a similar manner with the others; and indeed substantially the same methods might be applied if the generatrices of the helicoids were curved, as they sometimes are: in which case, however, the surfaces are no longer warped, but are of double curvature.

CHAPTER VII.

Isometrical Drawing, Cavalier Projection, and Pseudo-Perspective.

ISOMETRY.

329. In Fig. 255, *C* is a top view of a cube so placed that in the front view *A* the diagonals *cg*, *ab*, of its upper face are respectively parallel and perpendicular to the paper. The cube is shown

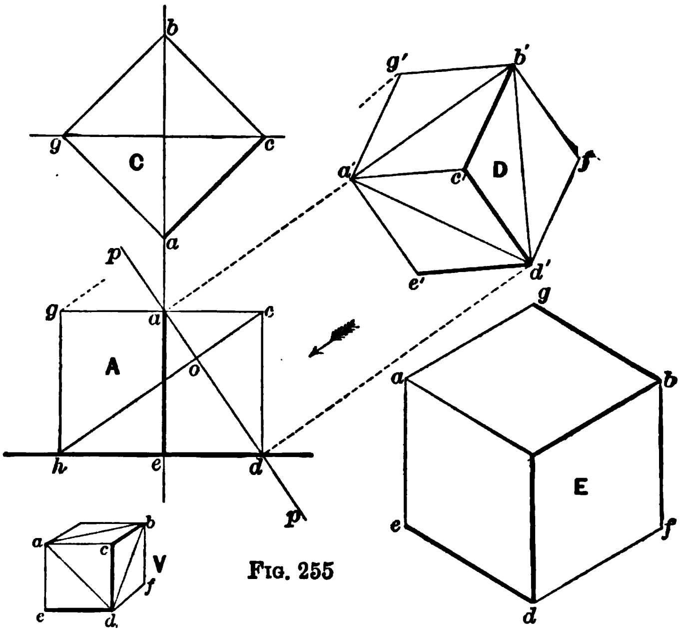


Fig. 255

as cut by a plane *pp*, perpendicular to the paper in view *A*; the section thus made, as seen in the perspective sketch *V*, is bounded by the three face diagonals *ab*, *ad*, *bd*: it is, then, an equilateral triangle, to the plane of which the three equal edges *ca*, *cb*, *cd* are

equally inclined. And as seen in view A, this plane is perpendicular to the body diagonal ch of the cube, which pierces it at o.

In the view D, which is an **orthographic** projection upon the plane pp, the three face diagonals are seen in their true lengths, forming the equilateral triangle $a'b'd'$. Since the three edges which meet at c are equally inclined to the plane, they will be equally foreshortened: therefore c' is the centre of the triangle, $a'c'$, $b'c'$, $d'c'$ are equal to each other, and the three angles at c' are each equal to 120°.

Every other edge of the cube being equal and parallel to one of these three, each visible one will appear equal and parallel to one of those already drawn; thus the apparent contour of the entire cube will be a regular hexagon, the representation of each face being a rhombus.

Because the edges of the cube are thus foreshortened in the same proportion, so that they and all parallels to them may be measured by the same scale, such a view as D is called an **Isometric Projection**; $a'c'$, $b'c'$, $d'c'$, are called the *isometric axes;* the planes which they determine, and all planes parallel to them, are called *isometric planes;* and all lines parallel to the axes are called *isometric lines.*

330. Drawings made in this manner possess the advantage of conveying, in one view, ideas of the three dimensions, as do those made in perspective; and in many cases they exhibit the peculiarities of structure more clearly than ordinary plans, sections, and elevations. They are readily understood by those who are not familiar with common projections; and in making sketches this system is very useful.

Obviously, however, the advantages of isometry are more pronounced when the objects to be represented are bounded by right lines, of which the principal ones are parallel and perpendicular to each other. It is not well adapted for the general drawing of machinery, since it involves an unpleasant distortion, and also because in most cases the circles are projected as ellipses.

331. *Distinction between isometrical* **projection** *and isometrical* **drawing.** In Fig. 255 the actual length of the edge of the cube is cd; its apparent length in view D is $c'd'$, equal to od in view A.

Suppose *cd* to be one unit in length—an inch for example: then by taking *od* as a unit it is possible to construct an *isometric scale*, by which all the isometric lines in *D* might have been set off; and such a scale could be used in constructing any isometrical *projection*.

This is a matter of purely abstract, theoretical interest, and not of any practical use whatever. Since the isometric lines are all *equally* foreshortened, there is no reason why they should be represented as foreshortened at all. Consequently an **Isometric Drawing** of the given cube is made as shown at *E*, each edge being drawn of its true length. This is the method always adopted in practice, the scales in common use being alone employed. The man who should construct a true *projection*, and send it to the workman to be measured, by an isometric scale, would simply make a record of his own stupidity; he who should teach others to do so, would commit a blunder of much more serious importance. For, to use the words of another, "the value of isometry as a practical art lies in the applicability of common and known scales to the isometric lines." *

332. We proceed, then, just as in making ordinary working drawings, setting off the dimensions on those lines either "full size," or with the 3-inch scale, the 1½-inch scale, etc., as the case may require. Naturally, lines which are vertical are so represented; the other isometric lines are then drawn with great facility by the aid of the T-square and the triangle of 60° and 30°.

Figs. 256–261 are simple exercises, composed wholly of isometric lines, the construction being so obvious that no detailed explanation is required: the method of locating the foot of the cross in Fig. 257, and the mortise and the tenon in Fig. 261, by measuring along the lines *ca*, *cb*, or parallels to them, is sufficiently shown by the dotted lines.

It is to be distinctly understood that these figures are illustrations merely: the student is not to *copy* them, but to construct them or others of similar character, with such variations of dimensions, arrangement, or design as may be suggested by his ingenuity, which should be given full play.

* W. E. Worthen.

333. Shadow Lines. In making mechanical drawings, on any system of projection, those lines which, being the intersections of

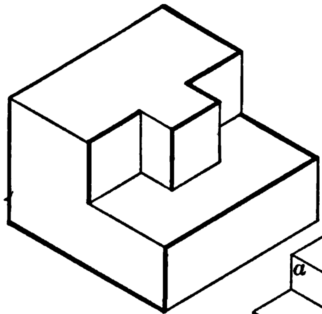


Fig. 256

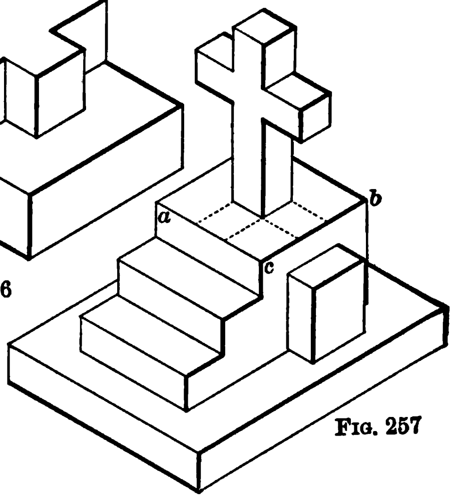
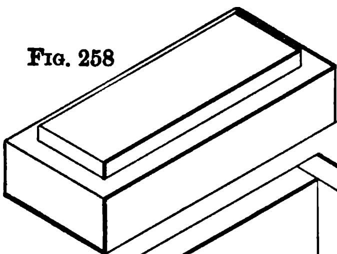


Fig. 257

surfaces which are illuminated with others which are not, intercept the light and thus cast shadows, are usually emphasized by making

Fig. 258



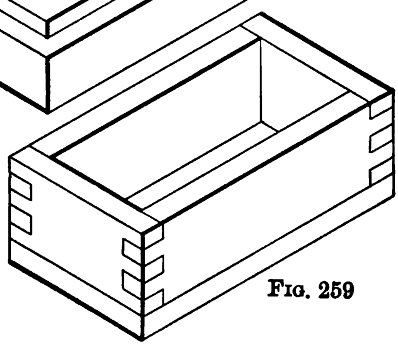


Fig. 259

them heavier than the other outlines. By this means an effect of relief is produced; the drawing is more easily read, and its appear-

ance greatly improved: and the lines thus emphasized are called **shadow lines.**

In orthographic drawings, the direction of the light is by common consent assumed as follows: Suppose the observer to be standing in a cubical room, facing one of the walls as the vertical plane; then the light comes from behind, the rays going downward to the right, in the direction of the body diagonal of the cube, each projection making an angle of 45° with the ground line.

And in isometric drawings reference is also made to the cube as a standard. Thus in *E*, Fig. 255, the light is supposed to have the direction of the body diagonal *af*, so that the faces *ec*, *cg*, are

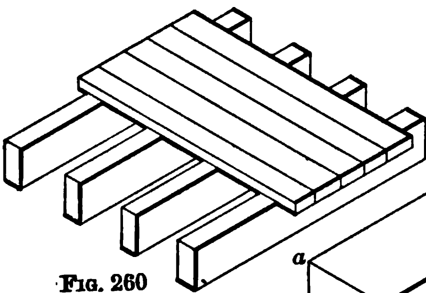


FIG. 260

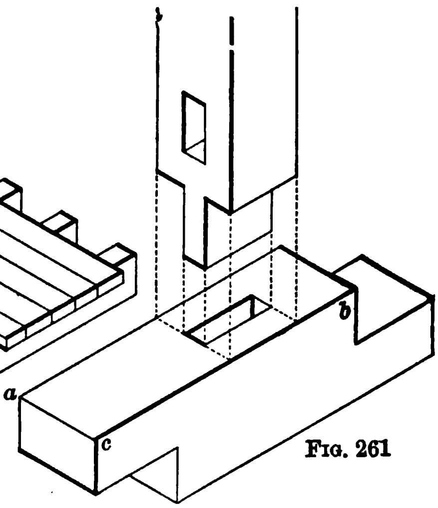


FIG. 261

illuminated, and shadows are cast by the edges *ed*, *dc*, *cb*, and *bg*. In drawing, the first of these lines should be made the heaviest, the last one the lightest, and the other two of equal and medium thickness.

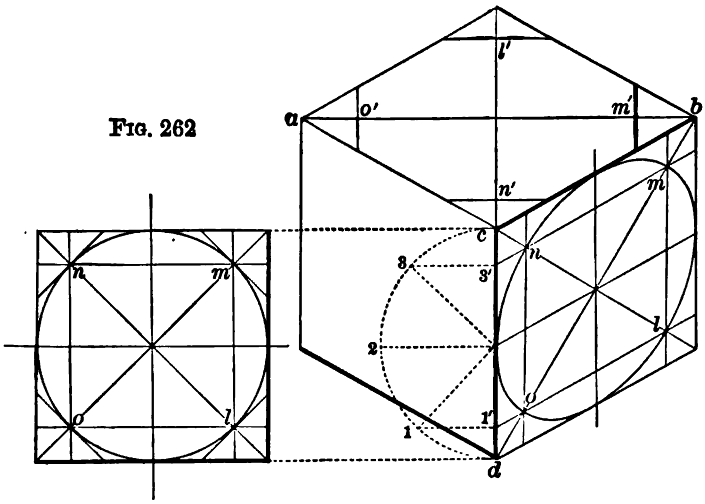
334. Isometrical Drawing of the Circle.—In the ellipse representing the circle inscribed in the face of the cube, Fig. 262, the axes coincide with the diagonals, and are at once determined by representing the parallels through *l*, *m*, *n*, *o*, in the elevation shown at the left. Describe a semicircle upon *cd* as a diameter, divide it into four equal parts by the points 1, 2, 3, draw 3 3′, 1 1′ perpendicular to *cd*, and through 3′ and 1′ draw parallels to *bc*; these

will cut the diagonals at m, o, and n, l, thus limiting the major and minor axes. As a check, note that lm and on should be parallel to cd.

The sides of the rhombus being equal, this construction may be made upon either one at pleasure. And, since all the faces of the cube are exactly alike, it follows that all circles lying in isometric planes are represented by similar ellipses.

By drawing tangents at the points l, m, n, o in the elevation the circle is circumscribed by a regular octagon, the isometric representation of which is therefore made by drawing at the corresponding

FIG. 262



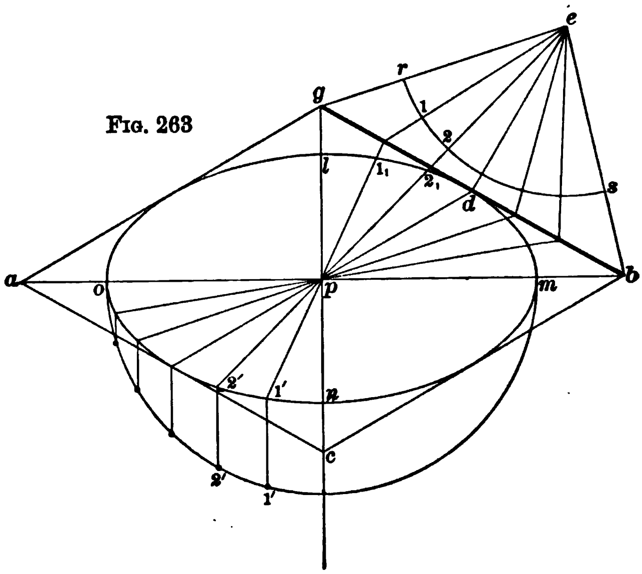
points l', m', n', o', in the upper face of the cube, perpendiculars to the diagonals, terminating in the sides of the rhombus.

335. Graduation of the Isometric Circle. *First Method.* At the middle point d of gb, in Fig. 263, erect a perpendicular de equal to db, and draw eb, eg; about e as a centre describe with any radius the quadrant rs, divide it as desired by the points 1, 2, 3, etc., through which draw radii and produce them to cut gb. From these intersections with gb draw lines to p, the centre of the ellipse: these will cut its circumference in the required points of division, 1_1, 2_1, etc.

Second Method. Describe a semicircle upon the major axis mo as a diameter, divide it in the desired manner by the points $1'$, $2'$,

etc., through which draw perpendiculars to *mo*, cutting the circumference of the ellipse in 1′, 2′, etc.: these will be the points required.

FIG. 263



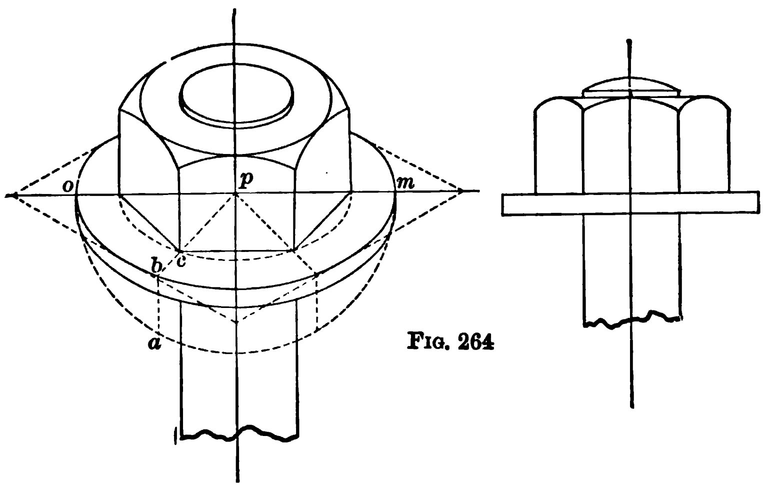


FIG. 264

An application of the above is shown in the drawing of the bolt, nut, and washer, Fig. 264. About *p*, the centre of the outer ellipse, describe an arc with radius po = semi-major axis, set off the

arc $oa = 60°$, erect the vertical ab, and draw bp cutting the inner ellipse (circumscribing the base of the nut) in c.

336. To draw Angles to the Sides of the Isometrical Cube (Fig. 265). Draw a square cg, whose side is equal to the edge of the cube; about one of its angles, say c, as a centre, describe the quadrant ab, graduate it, and produce the radii through the points of division to cut the sides of the square. The scale of tangents thus formed may, by cutting out the square, be applied to any side of the isometrical cube, thus determining the direction of a line in its face which shall represent a line making any required angle with its edge. For example, make $a'e' = ae$, and $b'f' = bf$: then $a'c'e'$,

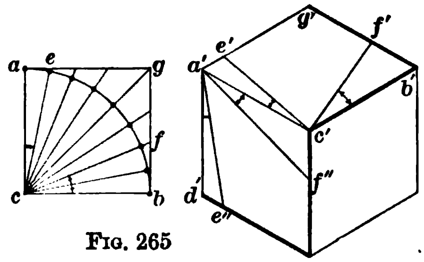


FIG. 265

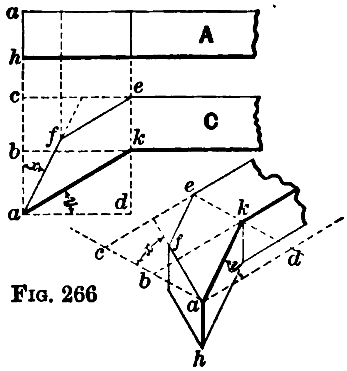


FIG. 266

$b'c'f'$ are the isometrical representations of the angles ace, bcf. The same angles are represented on the left-hand vertical face of the cube by making $d'e'' = ae$, $c'f'' = bf$, and drawing $a'e''$, $a'f''$.

An application of this is found in making the isometrical drawing of the piece shown in plan and elevation at C and A, Fig. 266, in which the angles x, y are assigned: also in C the distance ad is given, dke is perpendicular to ad, and ef parallel to ak: the thickness is uniform, and equal to ah in view A. The isometric drawing is lettered to correspond, and should require no further explanation.

337. Another method of dealing with lines which, though lying in isometric planes, are not parallel to either of the isometric axes, is by means of "offsets." Thus in Fig. 267 the slope of the diagonal brace is determined by measuring the distances cd, de,

along the isometric line *ca*, and setting up the vertical *ef*, of the values ascertained from the elevation shown on a reduced scale. This really amounts to the same thing, the angle being constructed by laying off the base and altitude of a triangle of which the required line is the hypothenuse,—which is in perhaps the majority of cases the most convenient means. Another illustration is given in the drawing of the box, Fig. 268; the outline of the end of the partially opened lid being set out by means of the vertical measurements *ca*, *co*, *cb*, *ch*, and the offsets *ae*, *ol*, *bd*, *hg*, taken directly

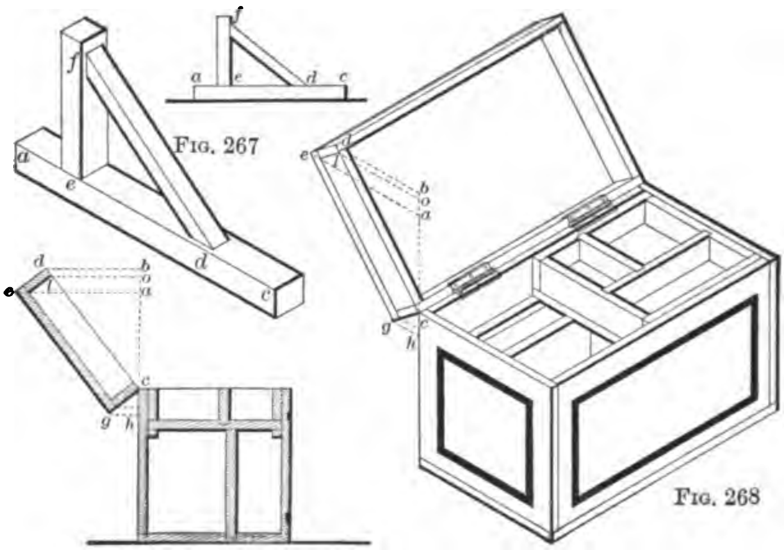


FIG. 267

FIG. 268

from the transverse section shown at the left.

338. This principle may be extended, and is applied to the determination of lines which do not lie in isometric planes; as illustrated in Fig. 269, representing the roof of a cottage, of the form and proportions shown in plan and elevations on a smaller scale at the right. The sloping lines of the roof at the nearer end are found by setting up the heights *ab*, *ad* on the vertical through *a*, and drawing the isometric lines *bh*, *df*: then the points *i*, *k* are the intersections of *bf*, *fh* with the isometrical line through *c*. A similar construction may be made at the farther end, thus fixing the line of the ridge *ff'*, on which the point *g* is located by setting off

fg, its distance from the plane *de*: we can then draw *gi*, *gk*, which do not lie in any isometric plane.

The same process is applied in drawing the wing roof, the heights *n*, *v*, *r*, being set up on the vertical through the nearer corner *m*, and the distance *oq* measured from the plane *mo*. The ridge line will pierce the main roof at a point *u*, which may be thus located: Set up $at = mr$, draw the isometric line *ts* cutting *bf* in *s*, and through *s* draw a parallel to *ff'*: this will cut the ridge line of the wing in the required point.

It will be observed that the lines *fh*, *gk*, and *ff'*, differ very

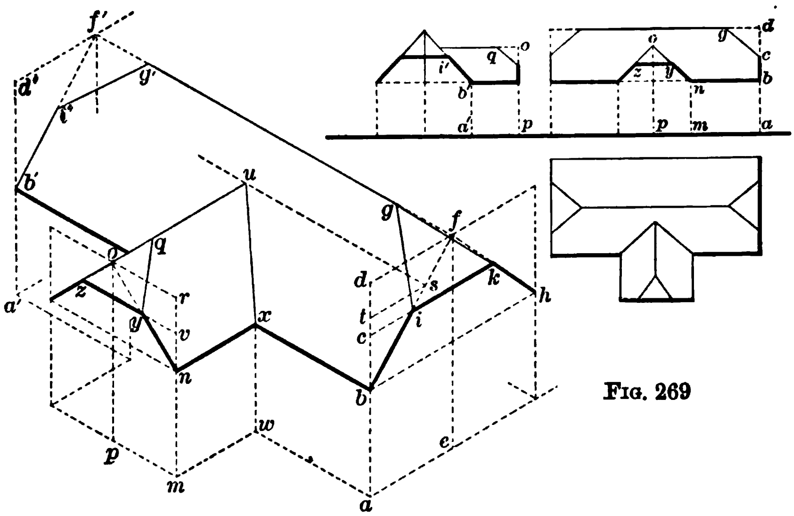


FIG. 269

little in direction, and *qz*, *ou*, differ still less. This simply shows that the farther side of the main roof is very nearly, and that of the wing roof almost exactly, perpendicular to the plane upon which the isometric drawing is made; and it will be perceived that in such cases this not a peculiarly eligible mode of representation, —as indeed it is not for architectural subjects of any description.

339. Thus far one of the isometric axes has been made vertical. But inasmuch as it is the relative direction of the lines among themselves which determines whether a drawing is an isometric one or not, there is no necessity that any of them should be vertical. In Fig. 270, for example, the principal lines are horizontal; but the drawings of the die and its matrix, and of the timber with its

mortises and its tenon, are at once recognized as isometric, and are just as easily understood as if they stood upright.

For convenience in constructing the drawings by means of the T-square and triangles, it is preferable in most cases, of course, that one of the isometric axes should be either vertical or horizontal, but should there be any reason for selecting other positions, there is nothing in the principle of isometry to prevent their adoption.

340. It will be noted that the correspondence of the die to the matrix in Fig. 270 is made much more obvious than it otherwise would be, by exhibiting the opposite ends of the two pieces. By

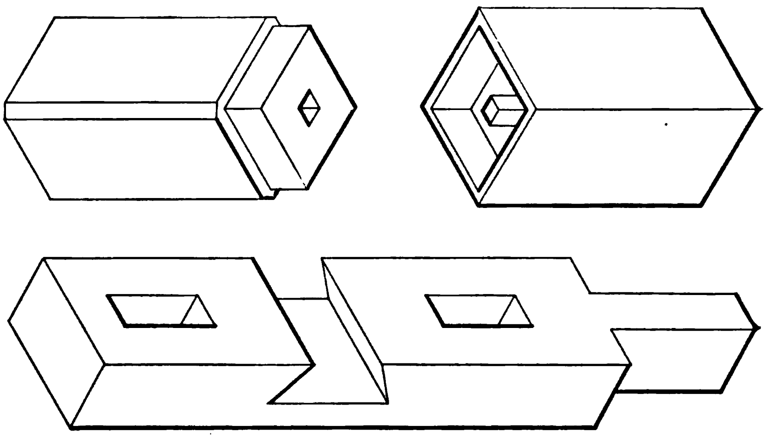


FIG. 270

merely turning the page around, it will be apparent that this could have been done equally well if the two pieces had been drawn in a vertical position.

For this reason isometry affords a means of illustrating in a very clear and striking manner many subjects in which views of the lower surfaces are desirable: a good example is shown in the drawing of the small shelf with its supporting bracket, Fig. 271.

In making such a drawing, as will readily be seen, the process is equivalent to constructing the projection of the cube, Fig. 255, upon the plane *pp*, as seen from the lower left-hand side, and looking in the direction opposite to that indicated by the arrow.

341. In Fig. 272 is shown an isometric drawing of the ratchet-wheel represented in the full-size views at the right. The backs of

the teeth not only terminate in, but are tangent to, the interior circle; and a test of the accuracy of the isometric drawing is found in the tangency of the edges to the ellipse representing that circle. And this embodies a principle capable of many other applications —as, for instance, in laying out a wheel with radial tapering arms:

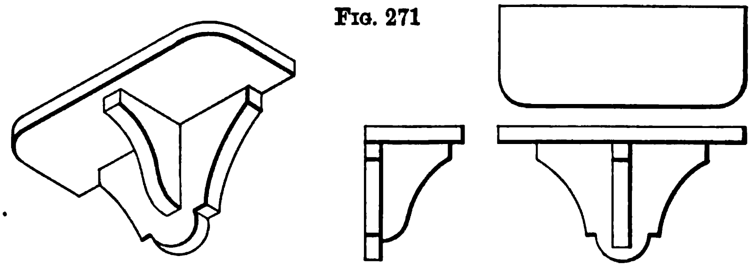


FIG. 271

the side outlines of each arm are tangent to a circle, which being drawn in the isometric construction, it is seen that the breadths of the arms at the outer ends only need be set out, thus fixing points through which tangents are to be drawn to the ellipse.

342. It seems needless to multiply examples, as it is believed that by the aid of the preceding any isometrical drawing likely to be required in practice may be constructed.

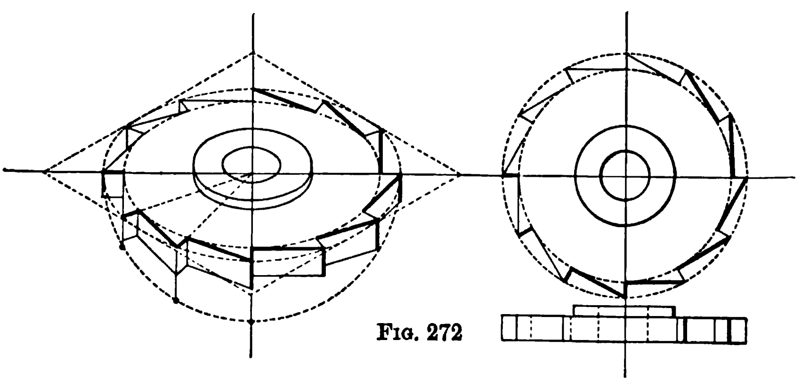
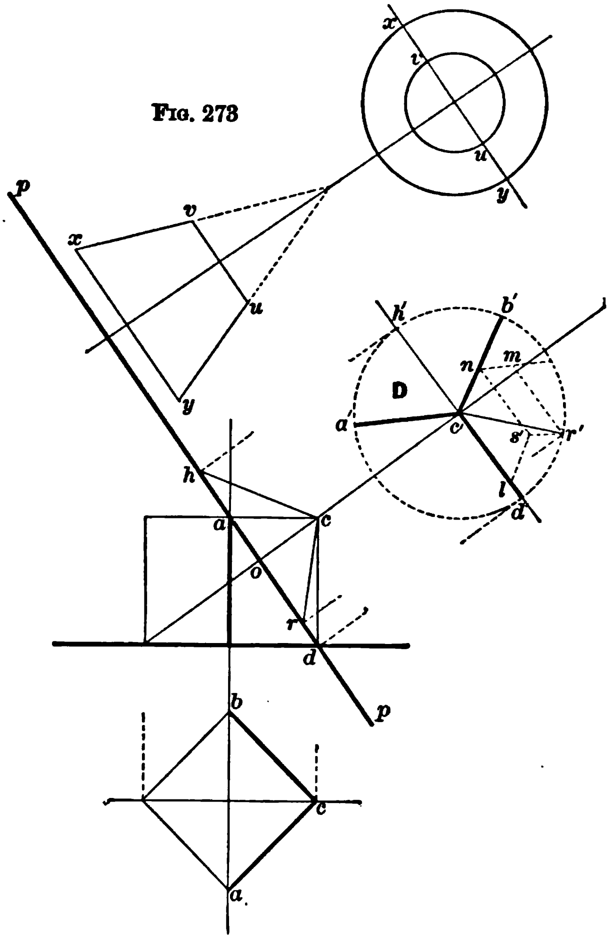


FIG. 272

In drawings of machinery, the circles of wheels, bearings, ends of shafts, and the like, will usually lie in isometric planes. Should occasion arise to represent one which does not, circumscribe it by a square: the projection of this will be a parallelogram, within which the ellipse may be drawn by any convenient method. So,

too, if it should be necessary to represent the section of a cylinder or a cone by an oblique plane, the solid may be conceived as surrounded by a square pyramid or prism, whose section by the given plane, as well as the isometric drawing of it, will be a parallelogram circumscribing the required ellipse.

FIG. 273



343. In conclusion, it may be pointed out that many lines not usually classed as *isometric* are strictly so in fact. This distinctive term is technically restricted to lines parallel to the isometric axes, which again are so called because they are equally foreshortened, and this is the result of their equal inclination to the plane upon

which they are projected. Now, in Fig. 273 the cube is cut by the plane *pp* as in Fig. 255, and in the projection *D* we at once recognize $c'a'$, $c'b'$, $c'd'$ as the isometric axes. If *cd* revolve around *co* as an axis it will generate a cone *dch*, all of whose elements make the same angle with the plane *pp*; so that any one of them, as *cr* (seen in *D* as $c'r'$), would be foreshortened in the same proportion as any other one. But in the isometric projection this fact would not be indicated by merely drawing $c'r'$: it is necessary to locate the point r' by means of offsets—$c'n$ giving its distance from the plane $a'c'd'$, $c'l$ its distance from the plane $a'c'b'$, and *nm* its distance from the plane $b'c'd'$. This being done, s' is at once seen to be the foot of the perpendicular $r's'$ from the point in question to the plane last mentioned.

Again, the isometric projection of any frustum of a cone, *xyuv*, whose bases are parallel to *pp*, would appear simply as two concentric circles, and without some auxiliary view that projection would convey no definite information about the cone, which might be of any altitude or have either base uppermost.

Since the whole value of isometry, in practice, lies in the power of imparting in one view definite ideas of the three dimensions, the above hints may serve a purpose as indicating possible relations of parts for the representation of which this method of drawing is not suitable.

CAVALIER PROJECTION.

344. In Fig. 274, let *MN* be a vertical glass plate representing the vertical plane; let *c* be a point in this plane, and *ca* a line perpendicular to it. Let *ar* be a visual ray, making an angle of 45° with the plane *MN*, and piercing it at *p*: then *cp* is the representation of *ca* upon the picture plane, and it is equal to *ca*, because the angles *cpa*, *cap* are each equal to 45°.

Suppose the eye to be at an infinite distance in the direction *ar*: then all the visual rays will be parallel, and all lines perpendicular to *MN* will be represented upon that plane by lines of their actual length, and parallel to *cp*.

The fact that the projection is of the same length as the perpendicular line *ca* depends upon the condition that the picture

plane cuts the projecting lines at an angle of 45°. But the direction of the projection depends upon that of the visual ray. Thus if the eye be still at an infinite distance, but in the direction *at*, the projection will have the direction *co*, but its length will remain equal to *ca*.

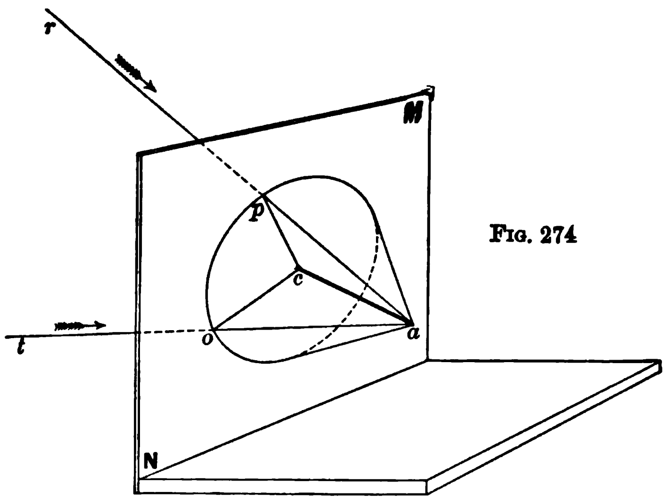


FIG. 274

Thus the direction of the projecting lines may be parallel to any element of the cone whose axis is *ca*, the angle at the vertex *a* being 90°, since all these elements make angles of 45° with the picture plane *MN*.

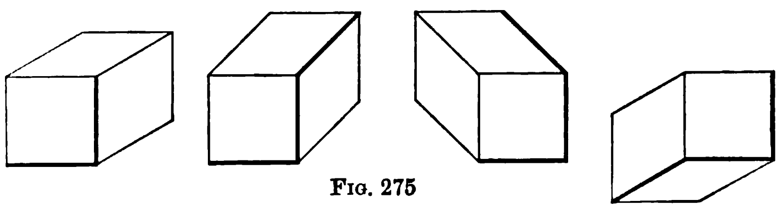


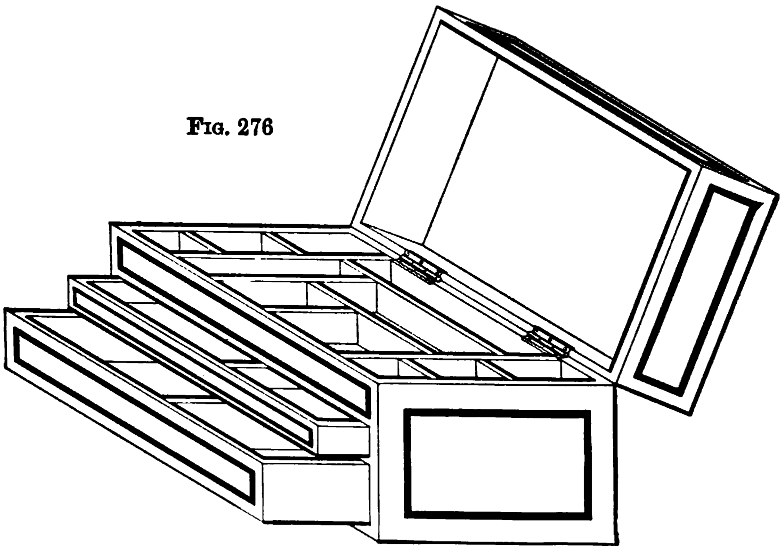
FIG. 275

345. Now any line which lies in the picture plane is its own projection. In representing a cube, therefore, as in Fig. 275, we may assume its nearer face to lie in that plane, and it will thus appear of its true form and size, that is, a square, as shown. From the preceding it follows at once that the edges which are perpendicular to *MN* may be represented by parallel lines of their true

length, but having any direction at pleasure; which enables us to show, in addition to the front face, either the right face or the left, the upper or the lower, as may best suit our purpose. And, as the figure shows, either of these faces at will may be made more conspicuous than the other by proper selection of the angles.

We have, then, a system of true *oblique projection:* it is more flexible than the isometric, always quite as easily executed and in many cases more so, and like it exhibits the three dimensions in one view. All lines lying in planes parallel to the paper are shown in their true forms and relations; and not only these but lines per-

FIG. 276



pendicular to the paper are shown of their actual dimensions, the introduction of any such senseless appliance as the "isometric scale" being prevented by the very nature of the process.

346. This system is well adapted for purposes similar to those in which isometric drawing is employed—such as the representation of joiner-work, as exemplified in the case of the box, Fig. 276, and in that of the peculiarly notched and fitted pieces shown in Fig. 277. In the illustration, and especially in the sketching of small mechanical details, it possesses the decided advantage over isometry that, as shown in Fig. 278, circles whose planes are

parallel to the paper are represented by circles, which greatly expedites the work of construction. Those lying in planes perpendicular to the paper, however, must here too be represented by ellipses: since each circumscribing square is projected as a rhombus, the axes will coincide with the diagonals, and may be found as in Fig. 262.

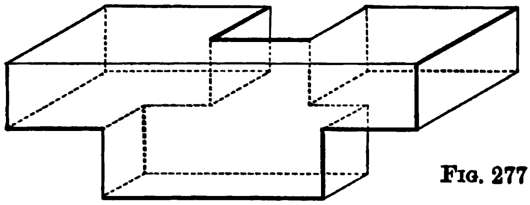
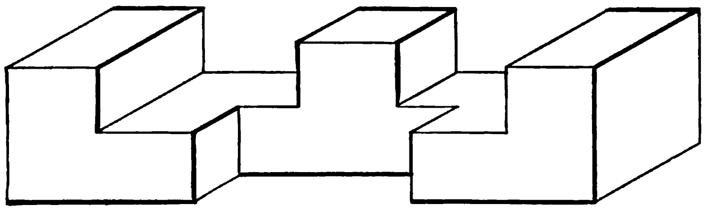


FIG. 277



The use of ordinates, or offsets, in determining lines which are neither parallel nor perpendicular to the paper is substantially the same as in isometric drawing. Thus in Fig. 279 the point *f* in the plane *cg* is located by setting off *gh*, its distance from the plane *gn*, and then *hf*, its distance from the front plane *gm*; the point *k* is

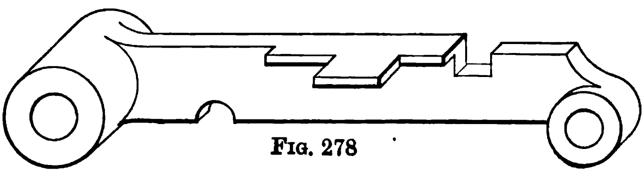


FIG. 278

determined by *ab*, its distance from the end plane *gn*, *bd* its height above the plane *cg*, and *dk* its distance in front of the rear plane, which is invisible. The two points *f* and *k* being thus fixed, the projection of the line *fk* is determined; and the rest of the construction can be readily traced without explanation.

The lines which cast shadows, and are therefore to be made

heavy, can usually be determined by inspection,—the light, as in the common orthographic projections, being supposed to come from over the left shoulder, and to go downward to the right as it recedes, as explained in (**333**).

347. There is, then, no need to pursue this subject farther: the principles which have been thus briefly set forth are sufficient for applying either of these modes of projection to any subjects within the common range of practice; and additional examples of cavalier projection are found in the pictorial illustrations introduced in the preceding chapters. Both are very useful, with certain limitations which have been suggested, and the question

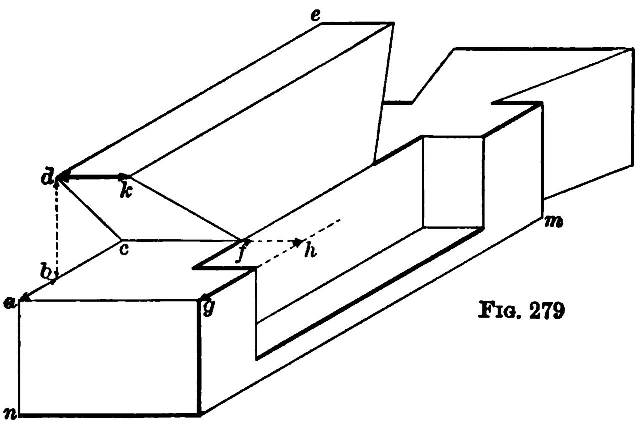


FIG. 279

whether either is suitable for any given case can be settled by experience alone.

But one thing has been decided by experience beyond all question; and that is, that the attempt to apply either isometric or cavalier drawing in the construction of a general plan of any complex machine is certain to result in a melancholy failure: the distortion, less noticeable in the case of minor details and detached pieces, becomes unendurable when the various parts are assembled.

PSEUDO-PERSPECTIVE.

348. For the purpose of producing a certain effect of relief, and conveying at least some idea of the three dimensions, while at the same time avoiding this distortion as well as the labor of con-

structing a true perspective drawing, a mode of representation has been devised, to which the name of Pseudo-Perspective seems appropriate, of which we give a single illustration in Fig. 280.

This is, in principle, a modification of cavalier projection; in that, as has been stated, the parallel projecting lines are inclined to the picture plane at an angle of 45°. But, referring to Fig. 274, it will be seen that if the cone of visual rays should have a less angle at the vertex, the projection of *ca* would be shorter; and by properly choosing this angle, the projection may be made shorter than the line in any desired ratio, while its direction is still entirely arbitrary.

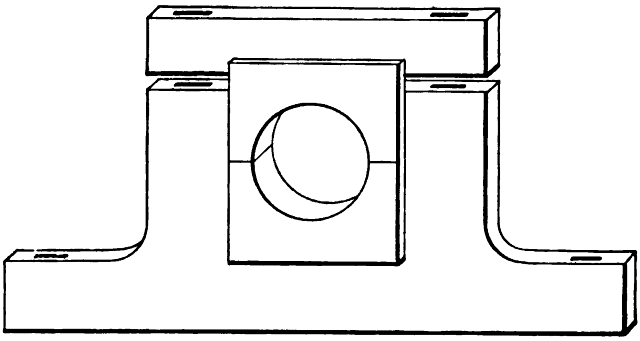


FIG. 280

The pseudo-perspective drawing, then, is made by representing the lines which are parallel to the paper in their true size, while those which are perpendicular to it are reduced to, say, one twelfth of the actual length, but parallel. This of course renders the result valueless as a working drawing; but it gives a sense of depth, and such representations are in many cases well suited for illustrations upon a small scale, such as cuts for encyclopædias and the like.

The distortion in the true cavalier projection is due to the mental recognition of the facts that the true representations of receding lines ought to converge, and that equal distances upon them ought to appear less as they recede. Both these errors are made less conspicuous by reducing the lengths of these representations, which is accomplished by the method of drawing above explained.

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